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Unsteady One-Dimensional Flow Over a Plane: Partial Equilibrium and Recession Hydrographs

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This paper presents a comparison of solutions of the shallow water (Saint Venant) equations for unsteady one-dimensional flow over a plane and solutions of the diffusion and kinematic wave equations, which are approximate forms of the Saint Venant equations. In the cases studied, the lateral inflow is constant and positive but may cease before a steady state flow is reached. It is shown that for highly subcritical flow, the criterion proposed by Woolhiser and Liggett is necessary but not sufficient to enable a choice between the shallow water equations and the kinematic approximation. An additional criterion is proposed for these cases.

INTRODUCTION

In 1967, Woolhiser and Liggett presented the first accurate, nondimensional solutions for the rising hydrograph for unsteady one-dimensional flow over a plane. The solutions were obtained partly by analytic techniques and partly by numerical methods utilizing the method of characteristics and the characteristic net. They demonstrated that as the parameter $k = S_0L_0/H_0F_0^2$ increases the solution to the continuity and momentum equations for the rising hydrograph approaches the solution for the simpler kinematic wave equation, where S_0 is the slope of the plane, L_0 is the length of the plane, H_0 is the normal depth for discharge $Q_0 = qL_0$ at the end of the plane with lateral inflow rate q, and F_0 is the Froude number with discharge Q_0 and depth H_0.

The parameter k, sometimes called the kinematic wave number, has frequently been used to judge when the kinematic wave equations are sufficiently accurate for hydrologic overland flow modelling.

Ponce et al. [1978] utilized a linear stability analysis of the Saint Venant equations to examine the applicability of the kinematic and diffusion models in open channel flow. Specifically, they compared wave attenuation factors and propagation celerities for these two models and concluded that most overland flow problems can be modelled as kinematic flow. Although their method does give considerable insight into the behavior of these models, it does not include lateral inflow nor does it include the relevant boundary conditions for overland flow. Therefore, it seems appropriate to utilize the approach of *Woolhiser and Liggett* [1967] and compare dimensionless overland flow hydrographs to develop approximate criteria.

The object of this paper is to examine partial equilibrium hydrographs (that is, hydrographs generated by lateral inflow that ceases before the system reaches a steady state) and recession hydrographs and to compare them with hydrographs obtained from the kinematic wave equation and the diffusion equation.

THE SHALLOW WATER EQUATIONS

The dimensionless shallow-water equations for flow over an impervious plane are

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$$\frac{\partial h_*}{\partial t_*} + u_*\frac{\partial h_*}{\partial x_*} + h_*\frac{\partial u_*}{\partial x_*} = R_* \tag{1}$$

$$\frac{\partial u_*}{\partial t_*} + u_*\frac{\partial u_*}{\partial x_*} + \frac{1}{F_0^2}\frac{\partial h_*}{\partial x_*} = k\left(1 - \frac{u_*^2}{h_*}\right) - \frac{R_*u_*}{h_*} \tag{2}$$

where the asterisks denote dimensionless variables defined as follows:

$$u_* = \frac{u}{V_0} \qquad h_* = \frac{h}{H_0} \qquad x_* = \frac{x}{L_0} \qquad t_* = t\frac{V_0}{L_0} \tag{3}$$

where u is the average velocity at some point x, h is the depth, and t is the time. The variables and parameters are indicated on the definition sketch (Figure 1). The dimensionless lateral inflow, R_*, is obtained by dividing the lateral inflow by the maximum rate, q_{max}. For a pulse input, R_* will take on the value of 1 when rain is occurring or 0 when rain ceases. In addition,

$$F_0 = \frac{V_0}{(gH_0)^{1/2}} \tag{4}$$

where V_0 is the normal velocity for $Q_0 = qL_0$. The friction slope has been defined by the Chezy equation, $S_f = u^2/C^2h$ which is reflected in the term u_*^2/h_* in the right-hand side of (2) and, of course, the numerical values of H_0 and V_0 for a particular case.

The initial conditions are $u_* = 0$, $h_* = 0$ at $t_* = 0$ for $0 \leq x_* \leq 1$, and the upper boundary condition is $u_* = 0$ at $x_* = 0$ for $0 \leq t_*$. The lower boundary condition is one of critical depth $u_* = (h_*)^{1/2}/F_0$ if flow is subcritical. No lower boundary condition is required for supercritical flow $u_* > (h_*)^{1/2}/F_0$.

Equations (1) and (2) reduce to a pair of ordinary differential equations on the domain designated zone A in Figure 2. Zone A is the domain enclosed by the line $t_* = 0$ and the forward and backward characteristic originating at the points $x_* = 0$, $t_* = 0$ and $x_* = 1$, $t_* = 0$, respectively.

The forward, or α, characteristic curve is described by the differential equation

$$\left(\frac{dx_*}{dt_*}\right)_\alpha = u_* + c_* \tag{5}$$

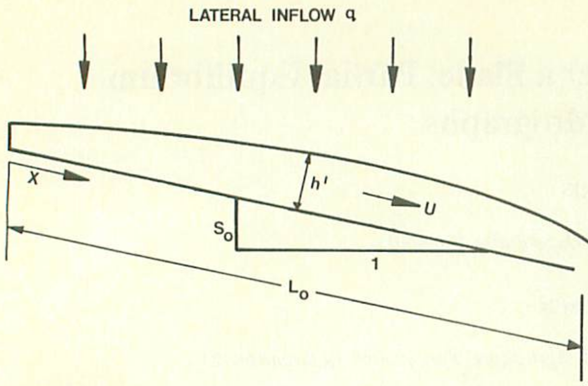

Fig. 1. Plane with lateral inflow.

and the backward, or β, characteristic is described by

$$\left(\frac{dx_*}{dt_*}\right)_\beta = u_* - c_* \tag{6}$$

where $c_* = (h_*)^{1/2}/F_0$. Equations (1) and (2) may be written in the normal form

$$\left\{(u_* + c_*)\frac{\partial}{\partial x_*} + \frac{\partial}{\partial t_*}\right\}\{(u_* + 2c_*)\} = k\left(1 - \frac{u_*^2}{h_*}\right) + \frac{R_*}{h_*}(c_* - u_*) \tag{7}$$

and

$$\left\{(u_* - c_*)\frac{\partial}{\partial x_*} + \frac{\partial}{\partial t_*}\right\}\{(u_* - 2c_*)\} = k\left(1 - \frac{u_*^2}{h_*}\right) - \frac{R_*}{h_*}(c_* + u_*) \tag{8}$$

The directional derivatives on the left-hand side of (7) and (8) are directed along the forward and backward characteristics (5) and (6)), respectively. Solutions to (1) and (2) can be obtained by analytic methods within zone A and by numerical integration of (7) and (8) along curves specified by (5) and (6), using methods described by *Liggett and Woolhiser* [1967].

PARTIAL EQUILIBRIUM

Partial equilibrium hydrographs may be obtained by the same techniques. If the dimensionless duration of rainfall, D_*, is smaller than $T_{\alpha\beta}$, which defines the maximum extent of zone A in time, (1) and (2) become

$$\partial h_*/\partial t_* = 0 \tag{9}$$

$$\partial u_*/\partial t_* = k[1 - (u_*^2/h_*)] \tag{10}$$

Equations (9) and (10) are valid within the domain enclosed by $t = D_*$ and the characteristics labelled α' and β' in Figure 2. The solutions to (9) and (10) are

$$h_* = h_0 \tag{11}$$

$$u_* = \frac{(h_0)^{1/2}\left[\exp\left\{\frac{2k}{(h_0)^{1/2}}(t_* - D_*)\right\} \frac{((h_0)^{1/2} - u_0)}{((h_0)^{1/2} + u_0)}\right]}{\left[\exp\left\{\frac{2k}{(h_0)^{1/2}}(t_* - D_*)\right\} + \frac{((h_0)^{1/2} - u_0)}{((h_0)^{1/2} + u_0)}\right]} \tag{12}$$

where u_0 and h_0 are the velocity and depth at time D_*. The expressions for the boundary characteristics α' and β' can be readily obtained by substituting the expressions for u_* and h_* given by (11) and (12) into (5) and (6) and integrating. The depth and velocity can be computed within zone A by using (11) and (12) after $t = D_*$ and the equations given by *Woolhiser and Liggett* [1967] before $t = D_*$. For the more general case of time varying lateral inflow the solutions presented by *Brutsaert* [1968] could be utilized in zone A before $t = D_*$. In the remainder of the solution domain, (5) to (8) are solved numerically using the methods described by *Liggett and Woolhiser* [1967].

Figure 3 shows partial equilibrium hydrographs for various values of F_0, k, and D_*. It is apparent that when the parameter F_0 is greater than about 0.4, the kinematic hydrograph provides an upper bound to the peak of the hydrograph for the shallow water equations (Figures 3*a*, and 3*c*). The greatest difference in the peaks occurs when $D_* = 1.0$. In general the Saint Venant equation hydrographs have a slower recession than the kinematic hydrographs. As k approaches a value of about 20, the kinematic hydrograph becomes a good approximation for $F_0 \gtrsim 0.4$.

THE DIFFUSION EQUATION

We have found that the numerical technique for solution of the full Saint Venant equations by the characteristic method, as described by *Liggett and Woolhiser* [1967], breaks down for low values of F_0 as well as for large k as they reported [*Woolhiser and Liggett*, 1967]. However, it is possible to obtain approximate solutions to the Saint Venant equations for small F_0 when k is large, and (2) may be written

$$\frac{\partial h_*}{\partial x_*} \approx F_0^2 k\left(1 - \frac{u_*^2}{h_*}\right) \tag{13}$$

Substitution from (13) into (1) gives the diffusion equation

$$\frac{\partial h_*}{\partial t_*} + \frac{\partial}{\partial x_*}\left\{h_*^{3/2}\left(1 - \frac{1}{F_0^2 k}\frac{\partial h_*}{\partial x_*}\right)^{1/2}\right\} = R_* \tag{14}$$

At the upstream boundary the condition $u_* = 0$ is equivalent to $\partial h_*/\partial x_* = F_0^2 k$ from (13). The critical flow downstream boundary condition $u_* = (h_*)^{1/2}/F_0$ is equivalent to $\partial h_*/\partial x_* =$

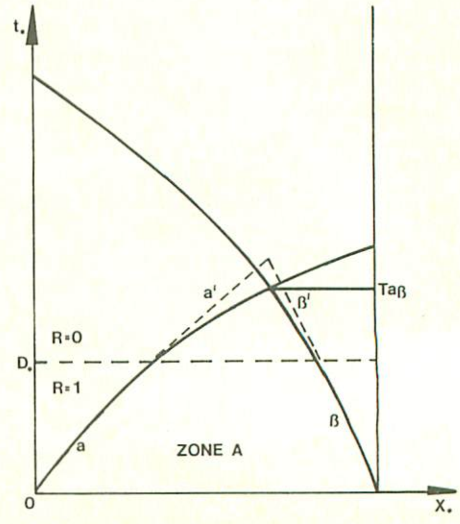

Fig. 2. The x_*-t_* plane.

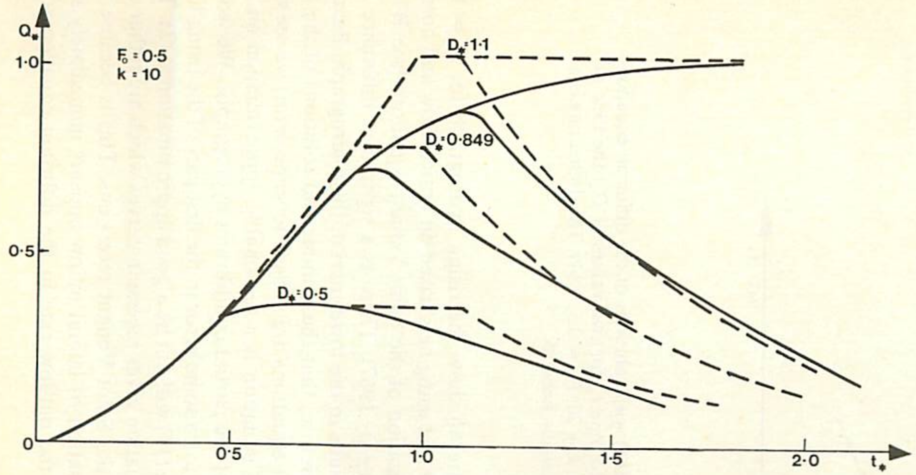

Fig. 3*a*

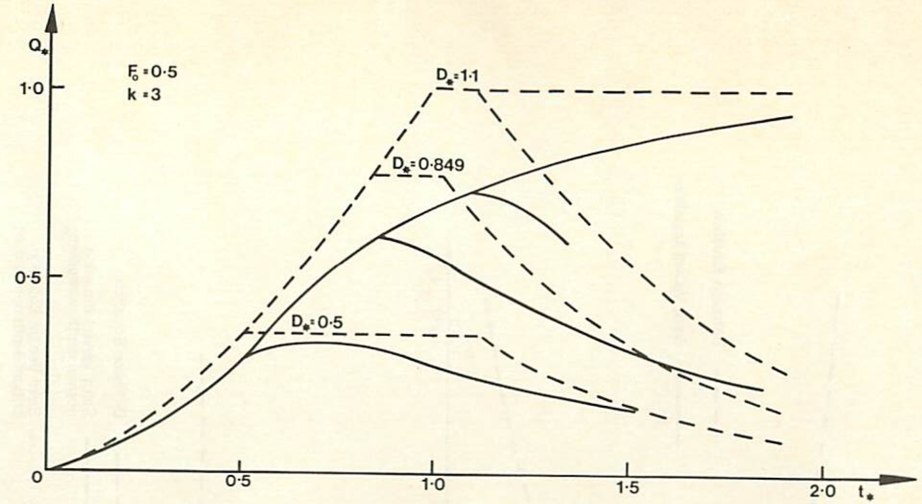

Fig. 3*b*

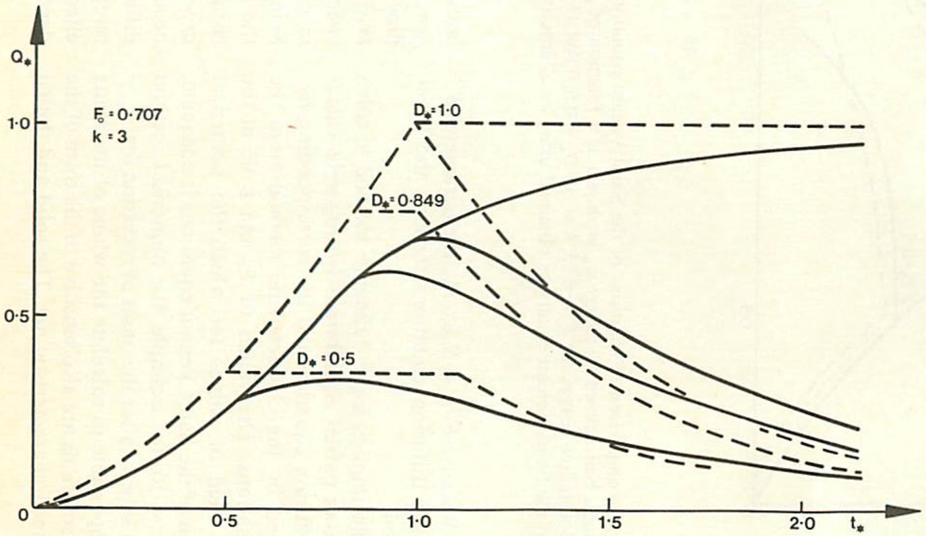

Fig. 3*c*

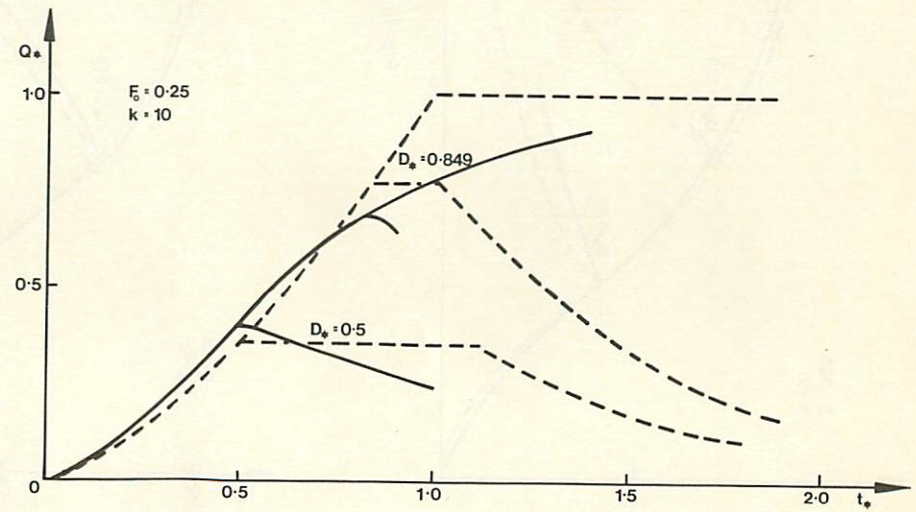

Fig. 3*d*

Fig. 3. A comparison of solutions of the Saint Venant equations (solid line) with solutions of the kinematic equation (dashed line). Normalized discharge q_* is shown as a function of normalized time t_* for various values of D_*, the time at which lateral inflow ceases. (*a*) $F_0 = 0.5$; $k = 10$; $D_* = 0.5, 0.849, 1.0$. (*b*) $F_0 = 0.5$; $k = 3$; $D_* = 0.5, 0.849, 1.1$. (*c*) $F_0 = 0.707$; $k = 3$; $D_* = 0.5, 0.849, 1.0$. (*d*) $F_0 = 0.25$; $k = 10$; $D_* = 0.5, 0.849$.

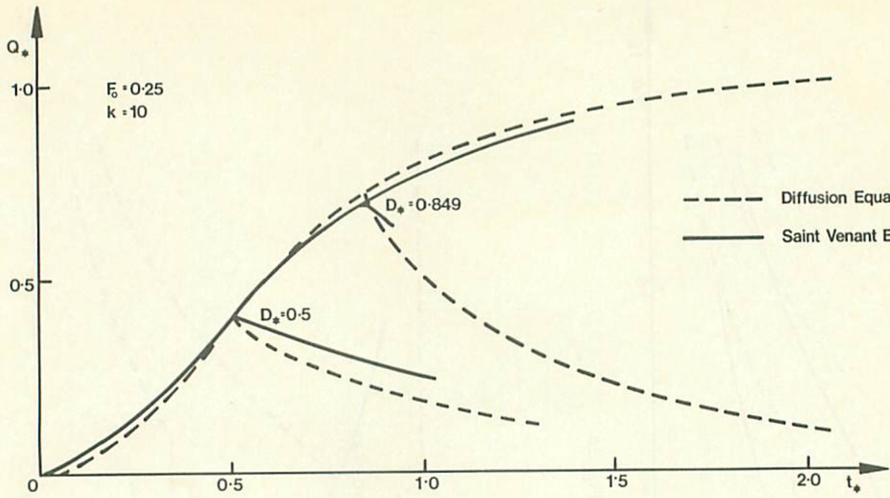

Fig. 4a

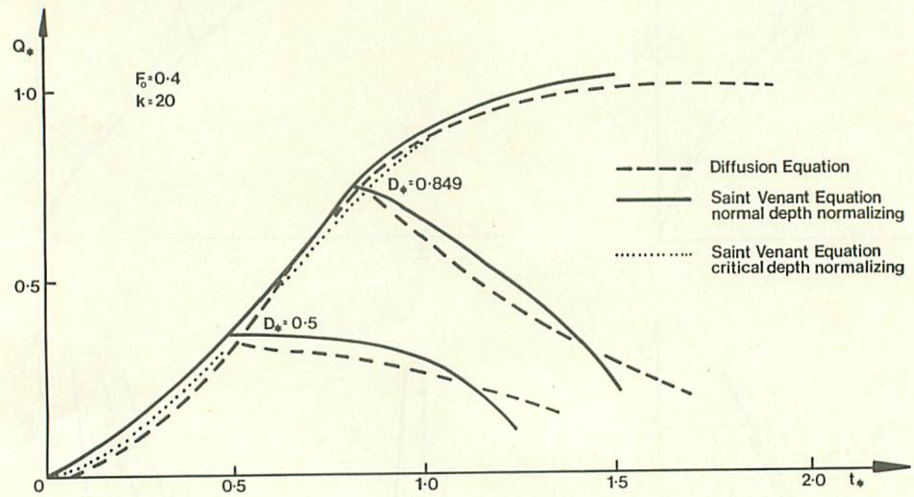

Fig. 4b

Fig. 4. A comparison of solutions of the Saint Venant equations (solid line) with solutions of the diffusion equation (dashed line). Normalized discharge q_* is shown as a function of normalized time t_* for two values of D_*, the time at which lateral inflow ceases. (*a*) $F_0 = 0.25$; $k = 10$; $D_* = 0.5, 0.849$. (*b*) $F_0 = 0.4$; $k = 20$; $D_* = 0.5, 0.849$. The dotted line is a solution of the Saint Venant equations obtained using an alternative characteristic network.

$F_0^2k[1 - (1/F_0^2)]$. Because $F_0^2k = S_0L_0/H_0$, the parameter F_0^2 only enters into the diffusion equations through the lower boundary condition.

We have used an implicit finite difference method to solve (14). Figure 4 shows partial equilibrium hydrographs calculated from the diffusion equation and the corresponding hydrographs obtained by the characteristic method from the Saint Venant equations. The values of F_0 and k lie at the boundary of the field of values for which the numerical method of solution of the Saint Venant equations is adequate. For $F_0 = 0.25$, $k = 10$, for example, the numerical method breaks down very shortly after the onset of recession for $D_* = 0.849$, and it is impossible to calculate the whole of the rising hydrograph. The solutions are also sensitive to the form of the numerically obtained characteristic net. The solid and dotted lines in Figure 4b show the rising hydrograph for $F_0 = 0.4$, $k = 20$ calculated using nets based on critical flow and normal flow normalization of the Saint Venant equations [see *Woolhiser and Liggett*, 1967]. There is a significant difference between the results in the lower part of the hydrograph. Bearing in mind, therefore, that these numerical solutions of the full Saint Venant equations are subject to some error, we see that the diffusion equation is a reasonable approximation on the rising limb of the partial equilibrium hydrographs. We would expect there to be some error in the first part of the rising limb because here (13) will not be a good approximation to (2). The diffusion equation gives recession curves which are rather different from the Saint Venant recessions. This is because the effect of cessation of lateral inflow appears immediately as a decrease in the outflow rate in the diffusion equation solu-

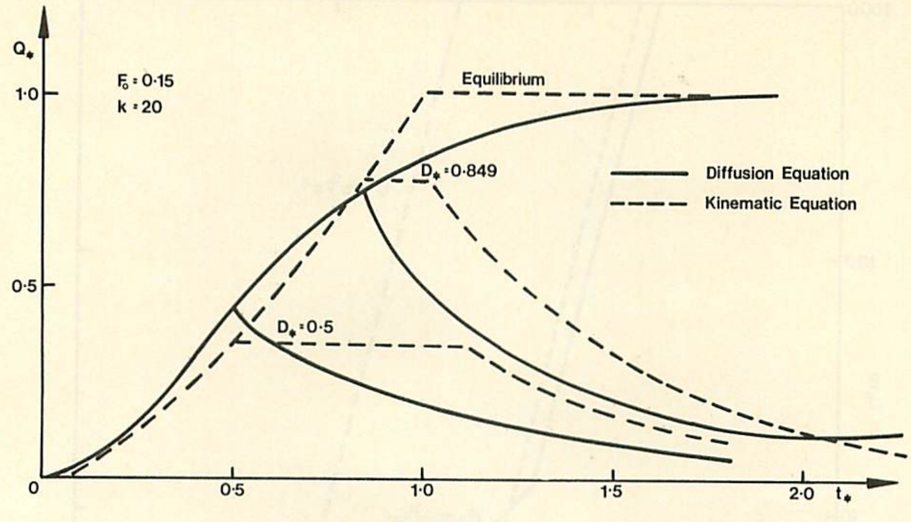

Fig. 5a

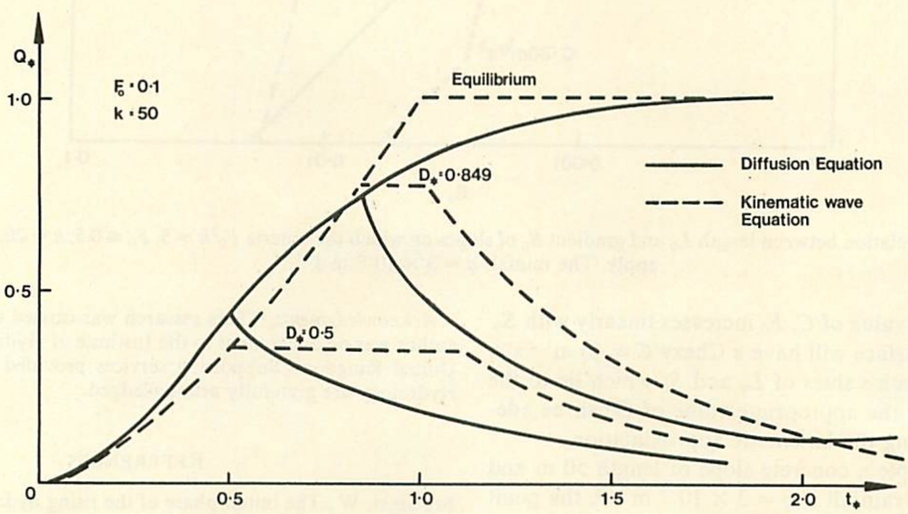

Fig. 5b

Fig. 5. A comparison of solutions of the diffusion equation (solid line) with solutions of the kinematic equation (dashed line). Normalized discharge q_* is shown as a function of normalized time t_* for various values of D_*, the time at which lateral inflow ceases. (*a*) $F_0 = 0.15$; $k = 20$; $D_* = 0.5, 0.849$. (*b*) $F_0 = 0.1$; $k = 50$; $D_* = 0.5, 0.849$.

tions, whereas with the full equations (and, for that matter, with the kinematic equations) there is a delay.

As $F_0 \rightarrow 0$ and $k \rightarrow \infty$, the diffusion equation solutions approach the Saint Venant solutions. Figure 5 shows partial equilibrium hydrographs calculated using the diffusion equation in the region of low F_0 and high k where the numerical solution of the characteristic equations fails. Fow low values of F_0 the kinematic equation is not a good approximation to the diffusion equation when $k = 20$. As $F_0 \rightarrow 0$, the value of k at which the kinematic equation can be used instead of the full shallow water equations increases approximately as $1/F_0^2$. This is because the downstream boundary condition has a marginal effect on the upstream depth profile when the kinematic approximation holds. The diffusion equation, which at low F_0 and high k is a good approximation to the shallow water equations, and its upstream boundary condition can be written in terms of one parameter F_0^2k. Thus when the downstream boundary condition, which includes both F_0^2k and F_0, is not significant, all solutions of the diffusion equation with the same F_0^2k are similar. For low values of F_0 the kinematic approximation can be made if $F_0^2k \gtrsim 5$. This condition is compatible with the condition $k > 20$ given by *Woolhiser and Liggett* [1967] for $F_0 \gtrsim 0.5$.

DISCUSSION

This result has interesting implications for the modelling of real overland flow problems. For a given roughness C and rainfall q the conditions $F_0^2k = 5$, $F_0 \leqslant 0.5$; $k = 20$, $F_0 \geqslant 0.5$ can be written as equations relating L_0, the length of a slope and S_0, its gradient. These equations are plotted in Figure 6 for a high rainfall, $q = 3 \times 10^{-5}$ m s^{-1} (about 4 in./h), and for three Chezy roughnesses. The change of gradient occurs at

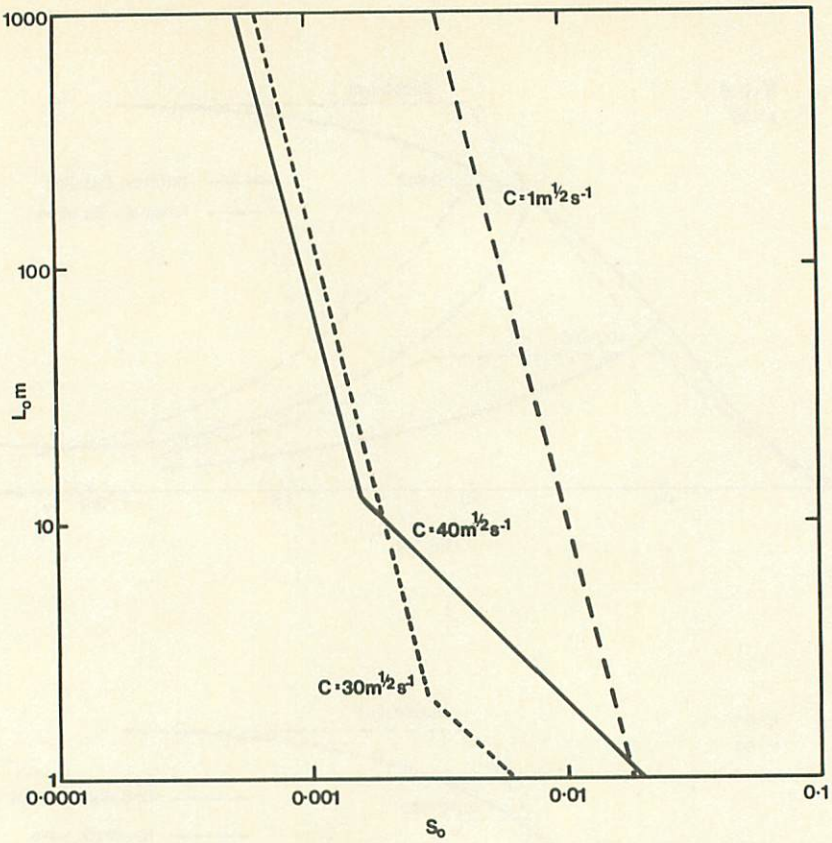

Fig. 6. The relation between length L_0 and gradient S_0 of slopes on which the criteria $F_0^2k = 5$, $F_0 \leq 0.5$; $k = 20$, $F_0 \geq 0.5$ apply. The rainfall $q = 3 \times 10^{-5}$ m s^{-1}.

$F_0 = 0.5$. For a given value of C, F_0 increases linearly with S_0. A smooth concrete surface will have a Chezy $C \simeq 40$ m$^{1/2}$ s^{-1}. The flow on slopes with values of L_0 and S_0 which lie to the right of the curve for the appropriate value of C can be adequately described using the kinematic approximation.

Consider for example a concrete slope of length 50 m and gradient 0.001. If the rainfall is $q = 3 \times 10^{-5}$ m s^{-1}, the point $L_0 = 50$ m, $S_0 = 0.001$ lies to the left of the curve for $C = 40$ m$^{1/2}$ s^{-1}. Thus the kinematic approximation should not then be used. Note that the new criterion $F_0^2k \geq 5$ for low values of F_0 restricts the range of values of L_0 and S_0 for which the kinematic approximation can be used. In particular, we suggest that it is necessary to use the full shallow-water equations, or at least the diffusion equation, for overland flow on very flat grassy slopes, although it has been assumed that since the values of k are very large the kinematic equation must be adequate.

Physically, the parameter F_0^2k or S_0L_0/H_0 represents the ratio of the difference in elevation between the top and bottom of the plane and the normal depth of flow at the downstream boundary. Therefore it is quite easy to decide whether or not the kinematic model is appropriate.

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