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Maximum Likelihood Estimation of Fourier Coefficients to Describe Seasonal Variations of Parameters in Stochastic Daily Precipitation Models

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ABSTRACT

Fourier series are convenient expressions for the seasonally fluctuating values of parameters in stochastic models of precipitation. Least-squares methods are often used to estimate the Fourier series coefficients, but this method has two important disadvantages. First the "data" points are in fact estimates of parameters, and because of varying sample size, they may have unequal variances and should not be given equal weight. Second, there is no statistically sound procedure to test the significance of individual harmonics.

In this paper we investigate methods to obtain maximum likelihood estimates of the Fourier coefficients to describe the seasonal variability in the parameters for a stochastic rainfall model. Parameters are obtained from a two-state Markov chain model for wet and dry day occurrence, and from a mixed exponential model for distribution of depth on wet days.

The procedure is demonstrated on four sample stations scattered across the continental United States. A constrained multivariate optimization scheme and a simple univariate procedure were used for maximum likelihood estimation, and these were compared with a least-squares estimate. The two seasonally varying parameters for the Markov chain are mutually independent, but the Fourier coefficients for each parameter are weakly dependent. The three seasonally varying parameters in the mixed exponential distribution are mutually dependent. However, for the four precipitation records analyzed, it appears that acceptable results can be obtained by simultaneously estimating the constant series terms and then independently estimating the harmonic amplitudes and phase angles. A likelihood ratio test can be used to test the significance of each added harmonic. It also appears from this analysis that the significant Fourier coefficients can be plotted on maps as isopleths, providing a concise description of regional precipitation climatology.

1. Introduction

In the past two decades many investigators have developed probabilistic models of daily precipitation. The seasonal variation in the parameters of these models has usually been accounted for by estimating the parameters for each m-week period ($m=1, 2, 4$) and tabulating them (Feyerherm and Bark, 1964). This approach requires 52 k m^{-1} individual tabular values for each station for a k-parameter model. In an attempt to be more parsimonious with respect to the number of parameters needed to describe precipitation at a particular location during a climatological year, several investigators have used Fourier series to describe the periodic seasonal fluctuations of parameters. Horn and Bryson (1960), Sabbagh and Bryson (1962) and Fitzpatrick (1964) used least-squares estimates of Fourier coefficients to examine the seasonal variations in mean monthly precipitation in the United States, Canada and Australia, respectively. Feyerherm and Bark (1965) and Fitzpatrick and Krishnan (1967) used Fourier series to account for parameter variation in first-order Markov chain models of precipitation occurrence. Ison *et al.* (1971) used least-squares estimates of Fourier coefficients to examine the seasonal variability of gamma distribution parameters for the amount of precipitation for an i day wet period ($i=1, 2, \ldots, I$). Woolhiser *et al.* (1973) used similar procedures to account for time variations in the parameters of a first-order Markov chain describing precipitation occurrence and also the parameters of the distribution function of the amount of precipitation.

Least-squares estimates of Fourier coefficients in this application are undesirable for two reasons:

1) The data points represent estimates of parameters and these estimates have unequal variances because of varying sample size or properties of the distributions

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being fitted. The least-squares procedure incorrectly gives each estimate the same weight.

2) There is no statistically sound procedure to test the significance of individual harmonics.

The maximum likelihood procedure described in this paper overcomes both of these objections.

2. Defining the precipitation model

We assume that the amount of precipitation falling on a day t is a random variable $Z_t = X_t \cdot Y_t$, the product of two independent random variables. X_t is described as a two-state Markov chain, taking on values of 0 or 1 depending on whether day t is dry or wet; Y_t is the amount of precipitation that falls on day t when $X_t = 1$. If the distribution of Y_t depends only on X_t and X_{t-1}, this is an example of a chain-dependent process (Katz, 1977). Y_t can be modeled in any one of a number of ways, depending on the causative meteorological process. As an example we will consider the case where Y_t is serially independent, is independent of X_{t-1}, and has a marginal distribution which is a mixture of two exponential distributions. As a generalization to allow for seasonal variability all of the parameters of X_t and Y_t may vary slowly with time over the year; we will describe each one of them with a finite number of terms from a Fourier series. We have found that the mixed exponential distribution describes daily precipitation significantly better than the simple exponential distribution. Furthermore, it appears that it may have some physical significance in locations where precipitation may occur from two different air masses.

To describe the periodicities in the precipitation process, we let the calendar day $n = (t \text{ modulo } 365) + 1$, $t = 1, 2, \ldots$. Let $\{X_t\}$ be a Markov chain defined by a transition probability matrix (tpm)

$$\mathbf{P}_n = \{\mathbf{p}_{ij}(n)\} = \{\mathbf{P}(X_n = j \mid X_{n-1} = i)\}\,;\; i, j = 0, 1.$$

The marginal distribution of X_t is given by the vector $\mathbf{p}(n) = \mathbf{P}_n \mathbf{p}(n-1)$, whose elements are both positive and sum to 1. Evidently, $\mathbf{p}(n)$ must satisfy the relation

$$\mathbf{p}(n) = [\mathbf{P}_n^T \mathbf{P}_{n-1}^T \cdots \mathbf{P}_1^T \mathbf{P}_{365}^T \cdots \mathbf{P}_{n+1}^T]\mathbf{p}(n) \quad \text{for } n = 1, 2, \ldots, 365,$$

where T indicates the transpose. Each $\mathbf{P}_n$ is defined by two parameters $p_{i0}(n)$, $i = 0, 1$, where $p_{i1}(n) = 1 - p_{i0}(n)$. $\{Y_t\}$ is assumed to be a stochastically independent sequence of random variables having a mixed exponential distribution whose probability density is given by

$$f_n(\lambda) = [\alpha(n)/\beta(n)] \exp\{-\lambda/\beta(n)\} + \{[1-\alpha(n)]/\gamma(n)\} \exp\{-\lambda/\gamma(n)\}, \quad (1)$$

where $0 < \beta(n) < \gamma(n)$ and $0 \leqslant \alpha(n) \leqslant 1$. Thus $\{Y_t\}$ is specified by three parameters, which may vary with the calendar day n.

The parameter set for the precipitation process can thus be written

$$\boldsymbol{\theta}(n) = \{p_{00}(n), p_{10}(n), \alpha(n), \beta(n), \gamma(n)\}. \quad (2)$$

Each parameter will be specified by a finite Fourier series, i.e.,

$$\theta_j = A_{j0} + \sum_{k=1}^{m_j} \left\{ A_{jk} \cos\left(\frac{nk}{T}\right) + B_{jk} \sin\left(\frac{nk}{T}\right) \right\}, \quad j = 1, 2, \ldots, 5, \quad (3)$$

where $T = 365/2\pi$ and m_j is the maximum number of harmonics needed to specify the parameter concerned. Thus a maximum of $2m_j + 1$ *coefficients* are needed to describe each *parameter* $\theta_j(n)$, which makes for parsimonious estimation.

An alternative (and equivalent) formulation of $\theta_j(n)$ which may save computational effort and lends itself more readily to regionalization by mapping, is

$$\theta_j(n) = C_{j0} + \sum_{k=1}^{m_j} \left\{ C_{jk} \sin\left(\frac{nk}{T} + D_{jk}\right) \right\}. \quad (4)$$

This form was used in the results described in Section 5.

3. Maximum likelihood equations

a. The Markov chain

At a station the observed sequence of wet and dry days $\{X_t\}$ is assumed to be exactly S years long. Thus $\mathbf{x}$ is a vector of length $N = 365\,S$ elements. Let $\boldsymbol{\phi}_1$ be a vector whose elements are the coefficients of the Fourier series describing $p_{00}(n)$ and $p_{10}(n)$. We wish to find the estimate $\hat{\boldsymbol{\phi}}_1$ of $\boldsymbol{\phi}_1$ which maximizes the likelihood function $L_1(\mathbf{X} = \mathbf{x} \mid \boldsymbol{\phi}_1)$. This function is specified by the number of transitions of the various types occurring on each of the calendar days $n = 1, 2, \ldots, 365$.

Let $a_{ij}(n)$ be the observed number of transitions from state $i(=0 \text{ or } 1)$ on day $n-1$ to state $j(=0 \text{ or } 1)$ on day n in the sample x. Clearly, $a_{ij}(n)$ can be any of the integers $0, 1, 2, \ldots, S$, and their ensemble is an array of size 365 by 4.

The logarithm of the likelihood function, which we will call U, can be written

$$U = \log L[\mathbf{X} = \mathbf{x} \mid \boldsymbol{\phi}_1] = \sum_{n=1}^{365} \{a_{00}(n) \log p_{00}(n) + a_{01}(n) \times \log[1 - p_{00}(n)] + a_{10}(n) \log p_{10}(n) + a_{11}(n) \log[1 - p_{10}(n)]\}. \quad (5)$$

U can be written as the sum of two subsidiary log-likelihoods U_0 and U_1 which are functions of $p_{00}(n)$ and $p_{10}(n)$, respectively. Thus the coefficients specifying $p_{00}(n)$ and $p_{10}(n)$ can be estimated independently if desired because $\partial U_j / \partial p_{i0}(n) = 0$ if $i \neq j$. Recalling that the $p_{ij}(n)$ are specified by the coefficients $\boldsymbol{\phi}_1$ and n, the estimation procedure requires that we maximize U with respect to ϕ_1. This is accomplished numerically, successively adding harmonics until a likelihood ratio test indicates no significant change in U. The details are discussed in Section 4.

In this example we have assumed that a first-order Markov chain is adequate to describe the precipitation process. Chin (1977) has shown that higher order chains are superior for many regions of the United States, particularly in the winter months. The approach described in this paper could also be applied to higher order Markov chains.

b. The mixed exponential model of $\{Y_t\}$

The observed S-year record of precipitation at a station can be written as a vector $\mathbf{z}$ of $N=365\,S$ elements which are either zero or positive. We wish to estimate the coefficients ϕ_2 which maximize the likelihood function $L_2(\mathbf{Y}=\mathbf{y}\,|\,\phi_2)$, where $\mathbf{y}$ is a vector consisting of the nonzero elements of $\mathbf{z}$. If we denote the logarithm of L_2 by U_2, then

$$U_2=\sum_{t=1}^{N} X_t \log f_n(z_t|\phi_2), \tag{6}$$

where, as before, $n=(t \text{ modulo } 365)+1$ and $X_t=0$ or 1 depending on whether z_t is zero or positive $f_n(z_t|\phi_2)$ is given by Eq. (1) in which $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ are specified by their Fourier coefficients ϕ_2 and n. This formulation ensures that only the nonzero elements of $\mathbf{z}$ contribute to U_2.

$\hat{\phi}_2$ is again found by maximizing U_2 numerically, but the elements of ϕ_2 are statistically dependent, which requires that they be estimated simultaneously if a rigorous approach is adopted. However, it was anticipated that the Fourier coefficients may be weakly cross correlated. If this were so, then independent estimation of the harmonics describing the various parameters, as compared with simultaneous estimation, would save considerable computational effort.

4. Numerical techniques for optimization

a. Optimization strategy

An examination of the derivatives of the likelihood function with respect to the coefficients in the Fourier series reveals that the harmonics are not independent. A rigorous approach to determining significant harmonics for p_{00} or p_{10} would therefore require calculation of the maximum likelihood function with a maximum of, for example, six harmonics for each parameter. This calculation would require multivariate optimization with 13 variables (the Fourier coefficients). Then six optimizations involving 11 coefficients could be carried out dropping out a different one of the six harmonics each time. The least significant harmonic of this set could be determined and a likelihood ratio test could be used to determine whether it should be dropped. This strategy could be continued until only the most important harmonics remain, but it would be very costly in computer time for the Markov chain parameters and even more so for the parameters of the mixed exponential because they are not independent. Therefore, a less rigorous approach was used. First the maximum likelihood estimates (MLE) of each parameter and the maximum likelihood functions U and U_2 were found, assuming that the parameters are constant throughout the year. Then the MLE of the amplitude and phase angle for the first harmonic was found and a likelihood ratio test was used to determine if the additional two parameters increased the likelihood function significantly. This procedure was followed for the second through the fifth harmonic and only those that were significant at the 0.01 level were retained.

b. Calculations

The polar form of the Fourier series as given by Eq. (4) was used in all optimization calculations. Starting values of the coefficients (means, amplitudes and phase angles) were obtained from least-squares fits to parameter values calculated for 14-day periods by numerical maximum likelihood methods.

The following three different numerical techniques were used to estimate the coefficients $\hat{\phi}_1$ and $\hat{\phi}_2$ by maximizing the log-likelihood functions U_0, U_1 and U_2:

(i) A multivariate, unconstrained optimization technique from the IMS (International Mathematical and Statistical) Library called ZXMIN.

(ii) A multivariate, constrained optimization technique using a modified Rosenbrock procedure called ROSEN.

(iii) A simple univariate scheme referred to as OPTI.

These techniques are described briefly in Appendix A.

To test the dependence between the parameters in the mixed-exponential distribution, we first estimated the phase angles and amplitudes of the first harmonic simultaneously using ROSEN and then estimated them individually using OPTI. The results are presented in Section 5.

5. Examples

National Weather Service daily rainfall records were obtained for Indianapolis, Indiana, Kansas City, Missouri, Tallahassee, Florida, and Sheridan, Wyoming, as examples. For each station, 20–25 years of data were used and the data were arranged so that calendar day 1 corresponds to 1 March. These stations were selected to include substantially different climatic conditions to test our optimization strategy and to determine the suitability and adaptability of the model.

The optimization results for the Markov chain are shown in Table 1. The rows labelled LS are the Fourier

TABLE 1. Coefficient estimates for the Markov chain.

Station	Parameter	Estimation method	C_0	C_1	D_1	C_2	D_2	U
Indianapolis	p_{10}	LS	0.537	0.0490	−1.831			
		ZXMIN	0.543	0.0566	−1.684			
		OPTI	0.537	0.0551	−1.680	NS*		
	p_{00}	LS	0.732	0.0762	−1.976	0.0253	1.249	
		ZXMIN	0.728	0.0669	−1.927	0.0221	0.825	−4467.5
		OPTI	0.732	0.0686	−1.932	0.0214	0.823	−4466.0
Kansas City	p_{10}	LS	0.579	0.0471	2.783			
		ZXMIN	0.586	0.0475	3.048			
		OPTI	0.579	0.0455	3.041	NS		
	p_{00}	LS	0.787	0.0742	−3.074			
		ZXMIN	0.783	0.0738	−2.974			
		OPTI	0.787	0.0723	−2.974	NS		−5067.6
Sheridan	p_{10}	LS	0.534	0.0655	−2.370			
		ZXMIN	0.544	0.0711	−2.258			
		OPTI	0.534	0.0686	−2.263	NS		
	p_{00}	LS	0.779	0.0603	−2.274			
		ZXMIN	0.775	0.0588	−2.123			−4170.0
		OPTI	0.779	0.0578	−2.122	NS		−4170.6
Tallahassee	p_{10}	LS	0.505	0.103	2.125	0.0479	−0.707	
		ZXMIN	0.536	0.113	2.383	0.0712	−0.299	
		OPTI	0.505	0.116	2.325	0.0572	−0.268	
	p_{00}	LS	0.761	0.0060	2.528	0.106	−0.578	
		ZXMIN	0.747	0.0568	2.680	0.102	−0.351	−4846.8
		OPTI	0.761	0.0490	2.912	0.0943	−0.347	−4855.8

* Not significant at 0.01 level by likelihood ratio test.

series coefficients obtained by least-squares fitting to parameter values estimated for 14-day periods. These coefficients were used as starting values for all optimization procedures. The values of C_0 for LS and OPTI are identical since both are, in fact, maximum likelihood estimates. The C_0 coefficient values for ZXMIN are different, reflecting the dependence between C_0 and the coefficients of higher harmonics. Coefficients are shown only for harmonics determined to be significant at the 0.01 level by a likelihood ratio test. Because our procedures require repeated use of the likelihood ratio test and the likelihood functions are "approximate" maxima, the effective significance level is somewhat different. The results of the significance tests were identical, except that OPTI declared the third harmonic of p_{00} significant for Tallahassee, apparently because the U value for two harmonics obtained by OPTI was 9 less than that attained by ZXMIN. A comparison of objective function values for the other three stations indicates that OPTI attained U values very close to those attained by ZXMIN. From this preliminary investigation it appears that the coefficients obtained by a simple univariate procedure result in values of the objective function acceptably close to those obtained by multivariate optimization. These coefficients were obtained by OPTI in from one-half to one-fourth of the computer time required by ZXMIN.

The coefficients' estimates for the mixed-exponential distribution obtained by least squares (LS), constrained multivariate optimization by the modified Rosenbrock Method (ROSEN) and a simple univariate procedure (OPTI) are shown in Table 2. In this example we were interested in examining the dependence structure between the coefficients and in comparing a sequential approximate optimization (OPTI) and a multivariate scheme (ROSEN). The strategy used was as follows: The ROSEN subroutine was called twice for each station; first to optimize the C_0 coefficients for α, β and γ (three parameters) and second to simultaneously estimate the C_0, C_1 and D_1 coefficients for α, β and γ (nine parameters). Two versions of OPTI were utilized. The first, used only for Indianapolis, used the C_0 values obtained by least squares and optimized the C_i, D_i, $i=1, 5$, coefficients in the following order: γ, β, α. After some preliminary runs we found that considerable improvement could be made in U_2 with little increase in computer time by taking into account the dependence between the C_0 coefficients. Therefore, the subroutine ROSEN was used to estimate the C_0 coefficients in the OPTI program. Univariate optimization was then used to estimate amplitudes and phase angles of the significant harmonics.

Examination of the objective function values U_2 in Table 2 reveals that the OPTI program performed

TABLE 2. Coefficient estimates for the mixed exponential distribution.

Station	Parameter	Estimation method	C_0	C_1	D_1	C_2	D_2	U_2	Number of coefficients
Indianapolis	α	LS	0.475	0.122	0.0897				
		ROSEN	0.451	0.0992	0.591				
		OPTI*	0.475	NS**					
		σ	0.018	0.0030	0.0030				
	β	LS	0.104	0.0682	−0.383				
		ROSEN	0.103	0.0593	−0.253				
		OPTI	0.104	0.0454	−0.580				
		σ	0.0034	0.0037	0.0043				
	γ	LS	0.529	0.122	−0.628				
		ROSEN	0.513	0.0875	−0.526			536.16	9
		OPTI	0.529	NS				526.19	5
		σ	0.011	0.0030	0.0030				
Kansas City	α	LS	0.362	0.114	2.64				
		ROSEN	0.365	0.125	2.482				
		OPTI	0.341	0.0878	2.521				
	β	LS	0.0660	0.0130	−2.526				
		ROSEN	0.0621	0.00091	constraint				
		OPTI	0.0598	NS					
	γ	LS	0.519	0.177	−1.478				
		ROSEN	0.500	0.157	−1.091			338.5	9
		OPTI	0.527	0.214	−1.031			334.1	7
Sheridan	α	LS	0.552	0.127	1.265				
		ROSEN	0.634	0.108	1.766				
		OPTI	0.595	NS					
		σ	0.021	0.0018	0.0018				
	β	LS	0.0570	0.0218	0.592	0.0181	−0.252		
		ROSEN	0.0631	0.0123	0.338				
		OPTI	0.0554	NS		0.01537	−0.459		
		σ	0.0031	0.0018	0.0018				
	γ	LS	0.240	0.138	−0.145				
		ROSEN	0.266	0.144	−0.309			2283.22	9
		OPTI	0.2712	0.161	−0.552			2290.4	7
		σ	0.0061	0.0019	0.0018				
Tallahassee	α	LS	0.450	0.0104	1.824				
		ROSEN	0.448	0.0	constraint				
		OPTI	0.438	NS					
	β	LS	0.145	0.0177	1.774				
		ROSEN	0.143	0.00417	0.896				
		OPTI	0.138	NS					
	γ	LS	0.863	0.0784	−1.017	0.164	0.1036		
		ROSEN	0.858	0.01378	−1.424			−769.29	9
		OPTI	0.845	NS		0.140	0.454	−758.69	5

* OPTI utilized for Indianapolis used least squares C_0 values for α, β and γ.
** Not significant at 0.01 level by likelihood ratio test.

creditably when the C_0 coefficients were estimated simultaneously. A reasonably objective comparison can be made for Kansas City. Here OPTI attained a U_2 of 334.1 with seven parameters while ROSEN attained 338.5 with nine parameters. Note that C_1 for β is very small and that D_1 was the constrained value, demonstrating the insignificance of this harmonic. For Sheridan and Tallahassee, OPTI attained higher values of U_2 than ROSEN with fewer parameters, but this is an unfair comparison since ROSEN was not used on

TABLE 3. Correlation matrix, Sheridan, Wyoming.

	$C_{0\alpha}$	$C_{0\gamma}$	$C_{0\beta}$	$C_{1\alpha}$	$D_{1\alpha}$	$C_{1\gamma}$	$D_{1\gamma}$	$C_{1\beta}$	$D_{1\beta}$
$C_{0\alpha}$	1.0	−0.074	0.843	−0.186	−0.187	−0.154	−0.167	−0.072	−0.074
$C_{0\gamma}$		1.0	−0.203	−0.028	−0.027	−0.144	−0.025	−0.056	−0.055
$C_{0\beta}$			1.0	0.088	0.093	0.089	0.107	0.105	0.103
$C_{1\alpha}$				1.0	−0.416	−0.351	−0.395	−0.253	−0.253
$D_{1\alpha}$					1.0	−0.348	−0.394	−0.260	−0.262
$C_{1\gamma}$						1.0	−0.362	−0.257	−0.258
$D_{1\gamma}$							1.0	−0.227	−0.228
$C_{1\beta}$								1.0	−0.382
$D_{1\beta}$									1.0

higher than the first harmonic while OPTI tested up to five harmonics for significance. An interesting outcome of this study is that likelihood functions which included a harmonic with parameters estimated by the least-squares technique were often smaller than maxima of the corresponding likelihood functions without the harmonic. This is surprising because the harmonic frequently explained a substantial part (up to 15%) of the variance in a least-squares sense. This added parsimony derived from the maximum likelihood procedure was not anticipated, but is advantageous in keeping the sampling variance of the estimates to a minimum.

To obtain the approximate correlations between, and the sampling variance of, the coefficients, the Hessian matrix was estimated numerically after optimization by ROSEN. (The method used to compute the Hessian and the correlation matrices is described in Appendix B). The sampling standard deviations of the coefficients (which are comfortingly small!) are shown in Table 2 for Indianapolis and Sheridan in the rows labeled σ, while the approximate correlation matrix for the coefficients for Sheridan is shown in Table 3.

The strong correlation (0.843) between $C_{0\alpha}$ and $C_{0\beta}$ for Sheridan is mirrored by a corresponding value of 0.747 for Indianapolis. (The correlation matrices for the two stations are qualitatively similar in all other respects, suggesting that the form of the correlation structure for the coefficients of this model will be reasonably invariant from station to station.) By contrast, the C_0 terms and the C_1 terms are weakly correlated, with a maximum correlation coefficient of 0.187. To exploit this interrelationship, we used the hybrid optimization strategy previously described, *viz.*, 1) find the joint optimum of $C_{0\alpha}$, $C_{0\beta}$ and $C_{0\gamma}$ simultaneously, using ROSEN, then 2) find the phase and amplitude of each harmonic for individual parameters using OPTI. To support this strategy, it will be seen from Table 3 that the amplitudes and phase angles of individual harmonics are negatively correlated ($-0.416<\rho<-0.362$), while the correlations between the coefficients of different parameters are also negative ($-0.393<\rho<-0.227$) but not as large in magnitude. Nonetheless, comparison of the two strategies (hybrid ROSEN/OPTI and ROSEN) indicates that the relatively weak cross

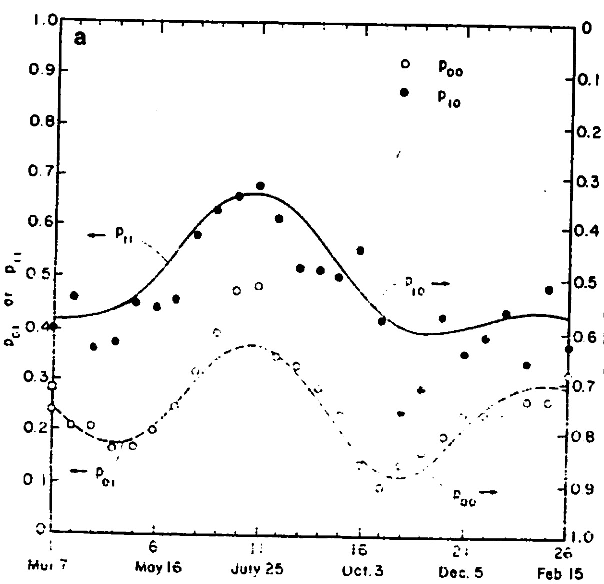

FIG. 1a. Markov chain parameters, Tallahassee, FL.

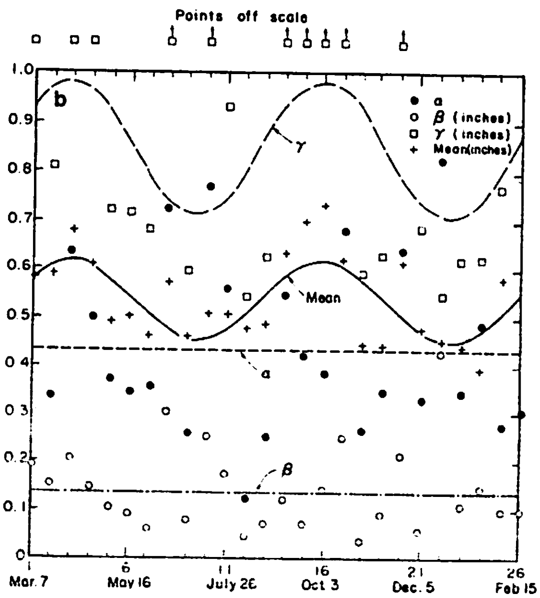

FIG. 1b. Mixed-exponential parameters, Tallahassee, FL.

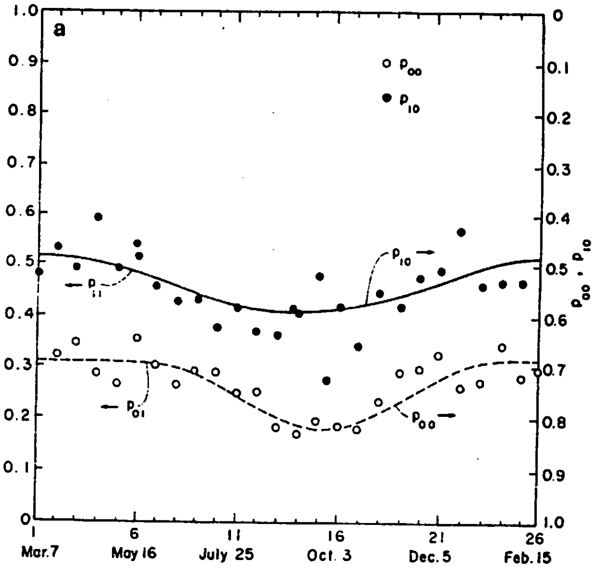

FIG. 2a. As in Fig. 1a except for Indianapolis, IN.

correlation allows one to obtain acceptable results with a fast, approximate method.

Parameter estimates for 14-day periods obtained by numerical maximum likelihood techniques for Tallahassee and Indianapolis are shown in Figs. 1 and 2. (It was the coefficients of the finite Fourier series fitted by least squares to these "data" points that were used as a starting value for the maximum likelihood procedures.) Plotted coaxially with these points are curves of the finite Fourier series of the parameters with coefficients determined by OPTI.

The Markov chain parameters p_{00} and p_{10} show a rather smooth seasonal variation for both of these stations with considerably greater amplitude in the fluctuations for Tallahassee. The first harmonics of p_{00} and p_{10} are very nearly in phase and this is also true

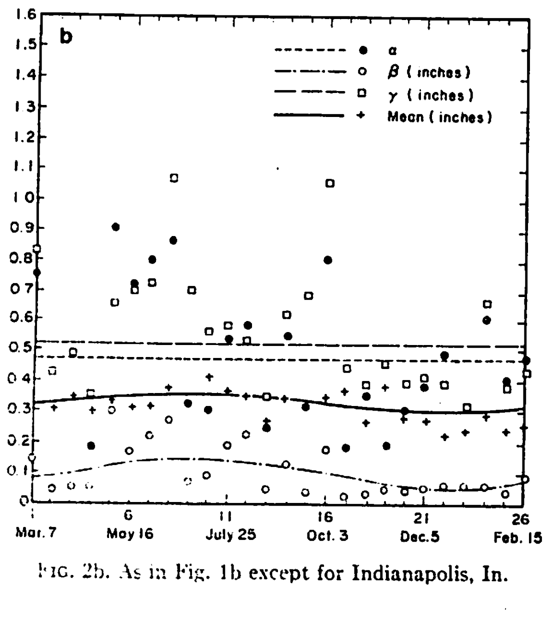

FIG. 2b. As in Fig. 1b except for Indianapolis, In.

for the other two stations. This observation might be utilized to further reduce the number of coefficients.

In contrast to the Markov chain parameters, the 14-day period estimates of parameters for the mixed-exponential distribution exhibit a great deal of random variation. Much of this variation can be attributed to the properties of the distribution. For example, as $\alpha \rightarrow 1$ the variance of the estimator of γ is unbounded. Similarly the variance of the estimator for β is unbounded as $\alpha \rightarrow 0$. Both stations had no significant harmonics for the weighting factor α at the 0.01 level. Tallahassee shows substantial fluctuations in γ with constant β, while Indianapolis has a constant γ with a low-amplitude fluctuation in β. Clearly, the distribution function for precipitation at Tallahassee has more seasonal variation than that at Indianapolis. The expected amount of rainfall, given that it occurs, at Tallahassee peaks around 3 October which is in the season of most frequent tropical storms. During this period the probability of rainfall is decreasing rapidly, reaching a minimum around 1 November. The high probability of rainfall and the high persistence of rainfall in late July apparently coincides with the peak of thunderstorm activity. Although rainfall is more likely during this period, the average amounts are smaller than in the tropical storm season and in late March and early April.

6. Strategy for parameter mapping

The parsimony attained by utilizing the finite Fourier series to describe the seasonal variations in parameters for this stochastic precipitation model is best illustrated by considering the examples from the previous section.

For each station and for the model used, 130 values of the five parameters were required for an adequate description in the twenty six 14-day periods, assuming constancy within those periods. By contrast when Fourier coefficients are used to describe the seasonal behavior of the five parameters, we found that an adequate description could be obtained by using from 13 to 17 coefficients depending on the station, and what is more, the value of the parameters can be evaluated for any calendar day. This description of the precipitation climatology requires only one to five more parameters than a listing of the monthly mean precipitation, but is much more informative.

Preliminary studies suggest that coefficients such as C_0, C_1 and D_1 can be plotted on maps and isopleths drawn, thereby presenting a regional and temporal picture of parameter variation. Coefficients could be interpolated for locations where there are no records (provided there were no significant topographic features) and equally likely sequences of daily precipitation could be generated by Monte Carlo techniques. This type of input data could be utilized effectively for daily hydrologic models. Coefficient maps could graphically portray rather subtle differences in precipitation characteristics and may be useful in conjunction with temperature and soils data in more precisely defining areas of plant adaptability.

It would be interesting to compare the seasonal variations in relative frequencies of different air masses at a point with seasonal variations in parameters of stochastic precipitation models, to determine if some sort of linkage can be established.

7. Summary and conclusions

Fourier series have frequently been used to concisely describe the periodic seasonal fluctuations of parameters in stochastic precipitation models. Previous investigators have estimated the parameters for daily, weekly or longer periods and have then fitted Fourier series to these estimates by standard least-squares techniques. This approach has to major deficiencies: 1) The data points are estimates and have unequal variances because of varying sample size from period to period or properties of the distribution being fitted. The least-squares procedure does not account for this. 2) There is no statistically sound procedure to test the significance of individual harmonics.

In this paper we investigate methods whereby maximum likelihood estimates of the Fourier coefficients describing seasonal variability in parameters of a stochastic precipitation model can be obtained by numerical optimization techniques. The stochastic precipitation model used as an example utilizes a two-state Markov chain to describe the occurrence of wet and dry days and a mixed exponential distribution for the amount of precipitation if the day is wet.

The two seasonally varying parameters for the Markov chain are mutually independent, but the Fourier coefficients for each parameter are weakly dependent. The three seasonally varying parameters in the mixed-exponential distribution are mutually dependent. However for the four precipitation records analyzed, it appears that acceptable results can be obtained by estimating the constant terms simultaneously and then estimating the amplitude and phase angle for successively higher harmonics for each parameter independently. A likelihood ratio test can be utilized to test the significance of each harmonic. If it is significant by this test, it is dropped, leading to a very economic model. The likelihood ratio test could also be used to test a hypothesis concerning the equality of coefficients for nearby stations.

It appears that the significant Fourier coefficients can be plotted on maps and isopleths drawn to provide a concise description of regional precipitation climatology.

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APPENDIX A

Optimization Techniques

1. *Multivariate, unconstrained optimization*

The subroutine ZXMIN is based on the work of Fletcher (1972) and involves the evaluation of U_i as a function of ϕ_1. It is a quasi-Newton method where the Jacobian and Hessian matrices are obtained by finite-difference approximation and interation.

The optimization subroutine ZXMIN was used only to estimate the coefficients for parameters of the Markov chain because only in this case could the necessary constraints be incorporated without extensive program modification.

2. *Constrained multivariate optimization*

Because $p_{00}(n)$ and $p_{10}(n)$ cannot be negative or greater than 1, and $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ are also constrained as indicated in Eq. (1), the unconstrained optimization program ZXMIN could not be used without modifications. These modifications could be made easily for estimating the coefficients of the Markov chain parameters but would have been much more difficult for the mixed-exponential parameters. Therefore the constrained multivariate technique (the modified Rosenbrock method or ROSEN) was used to estimate the Fourier coefficients for the mixed-exponential parameters.

The Modified Rosenbrock Method program was obtained from R. M. Li of Colorado State University. It consists of the Rosenbrock method with the Palmer (1969) version of computing orthonormal directions and added bound constraints for each parameter.

3. *Univariate optimization*

A simple univariate optimization scheme OPTI was used as a rapid method of estimating MLE of Fourier series coefficients to compare with the more accurate, but more costly, multivariate methods.

In subroutine OPTI a relative optimum of the likelihood function is first found by incrementing the phase angle D_i until a positive increment in the likelihood function is followed by a negative increment. A parabola is then fitted through the three points and the apparent maximum obtained by finding the phase angle, where $\partial U_i/\partial D_k=0$. A similar procedure is then used to find a relative maximum by varying the amplitude C_k. The improved estimates of D_k and C_k are then used as starting values for a second iteration. This method has been found quite satisfactory if the starting values of the parameters are reasonably close to the optimal values. This is helped by the fact that only a weak dependence exists between the amplitude and phase angle of successive harmonics, as had been conjectured before beginning this study.

APPENDIX B

Numerical Approximations

It is well known (see, e.g., Kendall and Stuart, 1968) that the variance-covariance matrix V of a set of parameters θ_i, $i=1, 2, \ldots, n$, can be obtained from the Hessian H of the log-likelihood function U evaluated at its maximum; in fact,

$$\mathbf{H}=\frac{\partial^2 U}{\partial\theta_i\partial\theta_j},\quad \boldsymbol{\theta}=\hat{\boldsymbol{\theta}}\quad \text{and}\quad \mathbf{V}=|-\mathbf{H}|^{-1}. \qquad \text{(B1)}$$

The diagonal terms of V are $\sigma_{\theta_i}^2$, so if we define S as an n-square matrix with σ_{θ_i} on the diagonal and the remainder void, then the correlation matrix of $\boldsymbol{\theta}$ is

$$\mathbf{R}=\mathbf{S}^{-1}\mathbf{V}\mathbf{S}^{-1}.$$

To obtain **H**, we have to proceed numerically when the log-likelihood function U does not exist in closed form, as is the case in this study. The result is that **R** and **S** will be approximations (perhaps quite close) to their true equivalents.

To explain the ideas and to avoid an unnecessarily complicated notation, we deal with a function $f(x,y)$ of of two independent variables x and y.

Using the two-dimensional Taylor series expansion about (x,y) with equal increments in each direction, we have, using central differences, that

$$\frac{\partial^2 f}{\partial x^2}=[f(x-\Delta, y)-2f(x,y)+f(x+\Delta, y)]/\Delta^2+O(\Delta^2), \qquad \text{(B2)}$$

and similarly for $\partial^2 f/\partial y^2$. The cross-derivative evaluated by central differences is

$$\frac{\partial^2 f}{\partial x\partial y}=[f(x+\Delta, y+\Delta)+f(x-\Delta, y-\Delta)-f(x+\Delta, y-\Delta)-f(x-\Delta, y+\Delta)]/4\Delta^2+O(\Delta^2). \qquad \text{(B3)}$$

Notice that this requires evaluating the function at four new points. If, on the other hand, we use a forward difference approximation to the cross derivative, we need only one extra function evaluation, at the cost of a little accuracy if Δ is small, so that

$$\frac{\partial^2 f}{\partial x\partial y}=[f(x,y)+f(x+\Delta, y+\Delta)-f(x+\Delta, y)-f(x, y+\Delta)]/\Delta^2+O(\Delta). \qquad \text{(B4)}$$

This latter formulation was used for evaluating the off-diagonal terms of **H** in this study.

The computational savings are considerable (54 evaluations versus 162 for $n=9$), especially when each function evaluation takes appreciable time.

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