

A NONLINEAR KINEMATIC WAVE MODEL FOR WATERSHED SURFACE RUNOFF**

VIJAY P. SINGH and DAVID A. WOOLHISER

New Mexico Institute of Mining and Technology, Socorro, N.M. (U.S.A.)

USDA—ARS, CSU Engineering Research Center, Fort Collins, Colo. (U.S.A.)

(Received October 22, 1975; revised and accepted January 6, 1976)

ABSTRACT

Singh, V.P. and Woolhiser, D.A., 1976. A nonlinear kinematic wave model for watershed surface runoff. *J. Hydrol.*, 31: 221–243.

The kinematic flow on a converging surface is considered as a simple nonlinear model to describe watershed surface runoff. Analytical and numerical solutions to kinematic wave equations are presented. The converging overland-flow model has three parameters: two are geometric parameters, and one is a kinematic wave friction relationship parameter. The geometric parameters are determined from watershed topography, and a simple procedure is outlined for transforming the complex geometry of a natural watershed into an equivalent converging-section geometry. The kinematic wave friction relationship parameter is obtained by an optimization procedure. The model's utility is appraised by considering its application to several natural agricultural watersheds located in two geographically distinct regions. A regression analysis is used to correlate the friction relationship parameter with physically measurable characteristics of watershed physiography. Finally, the proposed model is compared with Nash's linear model.

INTRODUCTION

Surface runoff from a watershed is generally recognized as a nonlinear process. The several nonlinear approaches that have been used in analyzing watershed response can be grouped into two categories:

- (1) Systems approach (Agorocho and Orlob, 1961; Bidwell, 1971).
- (2) Hydrodynamic approach (Chow and Ben-Zvi, 1973; Woolhiser and Liggett, 1967).

The former approach develops input–output relationships without making any explicit assumptions regarding the internal structure of the system. The latter approach requires the assumption that certain general laws of physics hold and further requires a geometrical abstraction of the real-world phenomenon.

* Contribution from Agricultural Research Service, USDA, in cooperation with Colorado State University Experiment Station, Fort Collins, Colorado 80523, U.S.A.

The physical laws required in the hydrodynamic approach are the equations of continuity and momentum. One geometrical abstraction that has been made by several investigators is that a watershed may be represented by a network of overland-flow planes and channels. Woolhiser and Liggett (1967) have shown that the simplified hydrodynamic approach based on kinematic wave theory is applicable to most hydrologically significant cases of overland flow. Since its development (Lighthill and Whitham, 1955), several investigators have used the kinematic wave theory to model surface runoff from natural watersheds (Wooding, 1965a, b, 1966; Brakensiek, 1967a, b; Kibler and Woolhiser, 1970; Eagleson, 1972; Singh, 1974). Most recognized that natural drainage systems had such complex surface configurations that it was necessary to transform the complex configuration into a simpler one with a similar hydrologic response. Several alternate geometric representations (Wooding, 1965a; Brakensiek, 1967b; Harley et al., 1970; Kibler and Woolhiser, 1970) have been hypothesized involving varying degrees of abstraction. A complex geometry will represent the watershed geometry more accurately but it may have more geometric parameters. The question arises: can a simple geometry be found to adequately portray some important aspects of surface runoff dynamics? In studies of the response characteristics of the linearly converging section, Woolhiser (1969) suggested that such a geometry might be a useful abstraction of a watershed regardless of its complexity. In this paper our objective is to present analytical and numerical solutions for the surface-runoff hydrograph from a linearly converging section with kinematic flow and to test its usefulness as a model for predicting peak discharge.

DEVELOPMENT OF SURFACE RUNOFF MODEL

Representation of watershed geometry

The only perfect model of a watershed is, of course, the watershed itself. The objective of any mathematical model is to transform the natural geometry into a simpler geometry and yet retain a similar hydrologic response. A linearly converging section of a cone (Fig. 1) is proposed as a simple representation of a natural watershed. From this figure, apparently the converging section has four geometric parameters including L_0 , r , θ and S_0 , where L_0 is the length of the section; S_0 is the slope; r is a parameter related to the degree of convergence; and θ is the interior angle. Because of radial symmetry, θ does not affect the relative response characteristics; only the watershed area must be preserved and is therefore dependent on L_0 and r .

The converging-section geometry possesses some interesting features:

- (1) Its discrete analog is a system composed of a cascade of unequal non-linear reservoirs (a systems view).
- (2) Its response is similar to that of a cascade of planes of decreasing size.
- (3) The convergence may account for the concentration of runoff at the mouth of a natural watershed.

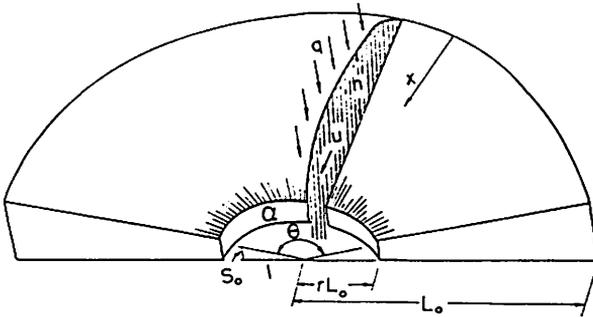


Fig. 1. Geometry of a converging section.

Nonlinear watershed dynamics and mathematical solutions

The dynamics of watershed response is described by the kinematic wave theory that includes the equation of continuity and an approximation of the equation of momentum. For unsteady, non-uniform flow over a converging section these equations are, the continuity equation:

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = q(x, t) + \frac{uh}{L_0 - x} \quad (1)$$

and the kinematic approximation to the momentum equation:

$$Q = \alpha h^n \quad (2)$$

or:

$$u = \alpha h^{n-1} \quad (3)$$

where h = local depth of flow; u = local mean velocity; Q = rate of outflow per unit width; $q(x, t)$ = rate of lateral inflow; x = space coordinate; t = time coordinate; and n and α are kinematic wave friction relationship parameters. The number of parameters can be reduced by normalization. The following normalizing quantities are introduced to make eqs. 1–3 dimensionless:

H_0 = normal depth for a discharge equal to the total steady-state outflow from the converging section divided by the mean width of the section

X_0 = $L_0(1-r)$, length of flow

$$T_0 = \left(\frac{1}{q_{\max}} \right)^{(n-1)/n} \left[\frac{L_0(1-r)}{\alpha} \right]^{1/n}$$

i.e., the time required by water to traverse the distance, X_0 , at velocity, V_0 , corresponding to the normalizing depth, H_0

$$V_0 = \alpha H_0^{n-1}$$

q_{\max} = maximum spatially uniform lateral inflow

$$q_0 = \frac{H_0 V_0}{L_0(1-r)}$$

$$Q_0 = \frac{H_0 V_0 (1-r)}{2r}$$

The derivation of normalizing quantities is given by Singh (1974). The dimensionless variables designated with asterisks are:

$$h_{\star} = \frac{h}{H_0}; \quad q_{\star} = \frac{q(x,t)}{q_0}; \quad t_{\star} = \frac{t}{T_0}; \quad x_{\star} = \frac{x}{X_0}; \quad u_{\star} = \frac{u}{V_0}; \quad Q_{\star} = \frac{Q}{Q_0}$$

The following equations are obtained when the dimensionless variables are substituted into eqs. 1–3:

$$\frac{\partial h_{\star}}{\partial t_{\star}} + u_{\star} \frac{\partial h_{\star}}{\partial x_{\star}} + h_{\star} \frac{\partial u_{\star}}{\partial x_{\star}} = q_{\star}(x,t) + \frac{(1-r) u_{\star} h_{\star}}{1-(1-r)x_{\star}} \quad (4)$$

$$Q_{\star} = h_{\star}^n \quad (5)$$

$$u_{\star} = h_{\star}^{n-1} \quad (6)$$

The parameter, α , no longer appears in the equations. Substituting eq. 6 into eq. 4 we obtain:

$$\frac{\partial h_{\star}}{\partial t_{\star}} + n h_{\star}^{n-1} \frac{\partial h_{\star}}{\partial x_{\star}} = q_{\star}(x,t) + \frac{(1-r) h_{\star}^n}{1-(1-r)x_{\star}} \quad (7)$$

which is a partial differential equation with a single family of characteristics. The system response can be examined by solving eq. 7, where the input, $q_{\star}(x,t)$, may be of three types: (1) pulse; (2) impulse; and (3) complex. The pulse input is a constant rate over some time interval; the impulse input assumes an instantaneous input of uniform depth; and the complex input consists of pulses of varying magnitudes. Analytical solutions are feasible only for the first two cases; numerical solutions are feasible for all three.

Analytical solutions. To obtain analytical solutions we assumed that the input is invariant in space. The method of characteristics can be used to obtain analytic solutions to eq. 7. The essence of the method when applied to wave motion is to find the space–time locus of discontinuity in the partial derivatives (with respect to time and space) of the dependent variable (depth

of flow). The locus defines the wave-path propagation. The differential equation of characteristic ground curve is:

$$\frac{dx}{dt} = n h^{n-1} \quad (8)$$

and the time derivative along the curve is:

$$\frac{dh}{dt} = q + \frac{(1-r) h^n}{1-(1-r)x} \quad (9)$$

The asterisks have been dropped in the development, but the variables are dimensionless unless noted otherwise. The solution to the system of eqs. 8 and 9 will be the solution to eq. 7.

Analytic solutions of eqs. 8 and 9 are presented in Appendix A. Analytic solutions are convenient when the input is of pulse or impulse type but become unwieldy for complex inputs. A second-order Lax-Wendroff scheme, derived in Appendix B, was used for numerical solutions. Combinations of analytic and numerical solutions or "hybrid solutions" saved computer time. The hybrid technique utilized analytic solutions when $q(x,t) = 0$ and at the downstream boundary.

Hydraulic resistance laws. If an approximation to the momentum equation is taken to be:

$$Q = C\sqrt{S_0} h^{1.5} \quad (10)$$

or:

$$Q = \frac{1.486}{n_M} \sqrt{S_0} h^{1.667} \quad (11)$$

Eq. 10 is the familiar Chezy's law and eq. 11 Manning's law, where C = Chezy's coefficient; n_M = Manning's coefficient; and S_0 = the energy slope. Comparing eq. 2 with eq. 10 we obtain:

$$n = 1.5 \quad (12)$$

$$\alpha = C\sqrt{S_0} \quad (13)$$

and comparing eq. 2 with eq. 11, we obtain:

$$n = 1.667 \quad (14)$$

$$\alpha = \frac{1.486}{n_M} \sqrt{S_0} \quad (15)$$

Eq. 10 will produce a constant Darcy-Weisbach friction factor.

APPLICATION OF MODEL TO NATURAL WATERSHEDS

For model calibration and testing, twenty-one natural watersheds were selected from two geographically distinct regions: five near Hastings, Nebraska, and sixteen near Riesel (Waco), Texas. These watersheds varied in area from 1.2 to 1,720 ha. Their detailed description can be found elsewhere (e.g., USDA (1963), and subsequent volumes). These watersheds were divided into two groups, one consisting of twelve terraced watersheds and the other consisting of eleven unterraced watersheds. If the area under terracing at any time was greater than or equal to 10% of the total watershed area, the watershed was considered to be terraced, otherwise unterraced.

Determination of mean areal rainfall

Rainfall-runoff data are available for these watersheds in the USDA publications on hydrologic data (e.g., USDA, 1963) which are released almost yearly, and consist of about one rainfall-runoff event per year for each watershed.

Although a watershed may have more than one raingage, data are normally available in the USDA publications for only a centrally located raingage, indicating that this represents the mean areal rainfall. For consistency this practice was followed on each watershed.

Determination of rainfall-excess

Rainfall-excess forms the lateral inflow, $q(t)$, and must be obtained by subtracting infiltration from precipitation. Philip's equation (Philip, 1957) was utilized to estimate infiltration loss. Philip's infiltration equation can be written as:

$$f = A + \frac{1}{2} S t^{-1/2} \quad (16)$$

where f = infiltration rate; t = time; and A and S are parameters dependent on soil characteristics and initial moisture conditions. Theoretically, these parameters will vary from storm to storm on the same watershed and from watershed to watershed for the same storm. Parameter A was considered roughly identical to the saturated hydraulic conductivity, thus it could be determined from physical characteristics of the soil. In the absence of information on hydraulic conductivity it was taken roughly as equal to somewhere between 50 and 80% of the lowest ϕ -index (for infiltration) estimated for several storms on the watershed under consideration.

Parameter S was allowed to vary with each rainfall episode, thus accounting for antecedent soil-moisture conditions. It was estimated for each storm by Newton's algorithm (Conte, 1965, pp. 19-65) subject to preserving the mass continuity. Parameter S was found to be sensitive to antecedent soil-moisture conditions and relatively less sensitive to A .

Parameter estimation

When the slope, S_0 , is incorporated into the parameter, α , the converging section has two geometric parameters: L_0 and r . If the area of the watershed is to be maintained θ is fixed when L_0 and r are known.

In an earlier study, Singh (1974) performed an optimization study where he showed that there was a strong interaction between the geometric parameter, r , and the resistance parameters. He found that a r value of 0.01 led to a minimum variance of α for several storms on the same watershed. He also found that the maximum straight line distance from the most remote point of the watershed to its outlet was a good estimator of the length of flow $L_0(1-r)$. Thus, a single measurement from the topographic map of a watershed is sufficient to transform the natural geometry into a simpler converging-section geometry.

Choice of objective function

The concept of determining optimal model parameters requires that the objective function be compatible with the intended use. However, there is difficulty in defining an error criterion that, upon minimization, will produce optimum parameter values without an undesirable bias.

The following objective function was used in this study:

$$F = \min \sum_{j=1}^M [Q_{p_o}(j) - Q_{p_e}(j)]^2 \quad (17)$$

where F = objective function; $Q_{p_o}(j)$ = observed hydrograph peak for the j^{th} event; $Q_{p_e}(j)$ = estimated hydrograph peak for the j^{th} event; and M = number of runoff events in the optimization set. This is particularly suitable for flood studies and seems to have some attractive features. Obviously greater weight is placed on higher peaks. If eq. 17 is divided by the number of events, the mean-squared error will result. This shows, on the average, the amount of error as the optimization is performed over a set of events. Because it requires only the hydrograph peak from each event, it is computationally efficient. A rather significant aspect is that it eliminates the effect of timing error resulting from improper synchronization between rainfall and runoff. The choice of this objective function was also determined by our interest in predicting hydrograph peaks. We can safely argue that if the model were a perfect representation of the system to be modeled, the choice of an objective function would have little or no effect on parameter values.

Parameter optimization

In a laboratory study of converging overland flow, Singh (1974) demon-

strated the reasonableness of keeping the kinematic wave parameter, n , fixed at 1.5, which would lead to less variance in the α .

If close geometric similarity between the mathematical model and the prototype watershed had been maintained, it would be appropriate to incorporate slope and a resistance parameter explicitly as in eqs. 10 and 11. However, geometrical distortion is so severe in representing natural watersheds by a converging section that optimized resistance and slope parameters would not have the same meaning as they do in eqs. 10 and 11. Therefore, α was chosen as the single parameter to be estimated. If a relationship can be found between α and measurable features of the watershed, the model may have predictive, as well as curve-fitting, capability.

A set of events was randomly selected from published data for each of the terraced and unterraced watersheds. However, published data consist of large events and are not randomly selected from all events. Utilizing the objective function of eq. 17, α was estimated for each watershed by a one-parameter optimization scheme designed on the principle of parabolic interpolation. To check the suitability of the objective function, hydrographs were reproduced for some events, using the optimized α . The reproductions were remarkably good both with respect to hydrograph peak and timing. This indicated that the objective function based on only hydrograph peak was reasonable, and that the kinematic wave model could reproduce the entire hydrograph well with the parameter optimized in the above manner.

Hydrograph prediction

A set of events different from those included in the optimization set was chosen for each watershed. Utilizing the optimized value of α , hydrograph predictions were performed for the events in the prediction set for each watershed. The model usually performed reasonably well. Predictions of hydrograph peak characteristics for two sample watersheds from terraced group and for two from unterraced group are given in Tables I and II. A sample of complete hydrograph prediction by the model is shown in Figs. 2 and 3. Although hydrograph timing was not involved in the procedure for estimating α , the timing was well predicted. This reflects favourably on the model and its underlying structure.

Correlation of parameter α with basin physiography

Investigating the physical significance of the parameter α may be interesting and useful. Naturally one might ask: can the parameter α be related to watershed characteristics that are physically measurable?

For each of the two groups of watersheds a simple regression analysis was performed, incorporating α as the dependent variable, and weighted slope, area, length of flow, and their various combinations as the independent variables. Weighted slope was computed by taking the weighted average of

TABLE I

Prediction of hydrograph peak characteristics by kinematic wave model on two terraced watersheds, Riesel (Waco), Texas

Watershed identification	Date of event	Observed hydrograph peak (cm/h)	Predicted hydrograph peak (cm/h)	Error in predicted hydrograph peak	Observed hydrograph peak time (min)	Predicted hydrograph peak time (min)	Error in predicted hydrograph peak
<i>Watershed Y-4</i> area = 32.3 ha $L_0(1-r) = 731.52$ m $r = 0.01$ $\alpha = 1.142$	04/24/1957	4.09	5.25	-0.284	48.0	49.3	-0.037
	05/13/1957	2.90	3.13	-0.019	47.0	50.0	-0.064
	06/04/1957	4.04	4.19	-0.036	37.0	44.7	-0.208
	03/29/1965	6.35	6.43	-0.013	60.0	86.9	-0.311
<i>Watershed Y-7</i> area = 16.2 ha $L_0(1-r) = 548.64$ m $r = 0.01$	04/24/1957	6.00	7.09	-0.181	31.0	30.9	0.002
	05/13/1957	5.16	5.56	-0.078	33.0	33.0	0.000
	06/23/1959	4.47	3.68	0.177	57.0	112.0	-0.964
	03/29/1965	5.17	7.12	-0.336	79.0	74.1	0.056

$$\text{error} = \frac{(\text{observed quantity} - \text{predicted quantity})}{\text{observed quantity}}$$

TABLE II

Prediction of hydrograph peak characteristics by kinematic wave model on two unterraced watersheds near Hastings, Nebraska

Watershed identification	Date of event	Observed hydrograph peak (cm/h)	Predicted hydrograph peak (cm/h)	Error in predicted hydrograph peak	Observed hydrograph peak time (min)	Predicted hydrograph peak time (min)	Error in predicted hydrograph peak time
<i>Watershed W-3</i> area = 194.67 ha $L_0(1-r) = 2298.2$ m $r = 0.01$ $\alpha = 3.124$	09/05/1946	1.40	1.41	-0.085	146.0	163.6	-0.121
	06/05/1949	0.36	0.26	0.266	74.0	97.0	-0.322
	06/25/1951	1.70	1.26	0.260	129.0	168.3	-0.304
	07/13/1952	3.38	3.99	-0.180	124.0	133.7	-0.079
	06/15/1947	2.49	2.41	0.034	64.0	66.7	-0.042
	06/16/1957	1.73	1.45	0.160	154.0	64.2	-0.067
<i>Watershed 4-H</i> area = 1.473 ha $L_0(1-r) = 152.4$ m $r = 0.01$ $\alpha = 6.43$	08/11/1939	4.52	6.87	-0.525	5.0	5.4	-0.071
	06/20/1942	5.82	6.37	-0.095	8.0	6.1	0.235
	09/05/1946	3.89	4.42	-0.137	12.0	6.4	0.466
	06/01/1951	6.16	8.34	-0.234	123.0	124.5	0.013
	07/13/1953	9.19	10.04	-0.092	21.0	5.3	0.750
	06/12/1958	0.92	1.22	-0.330	14.0	10.5	0.252
	06/12/1965	9.10	9.20	0.055	19.0	10.2	0.474
	06/12/1965	6.14	6.48	-0.056	7.0	11.0	-0.571
06/29/1965	8.10	10.98	-0.352	3.0	5.0	-0.667	

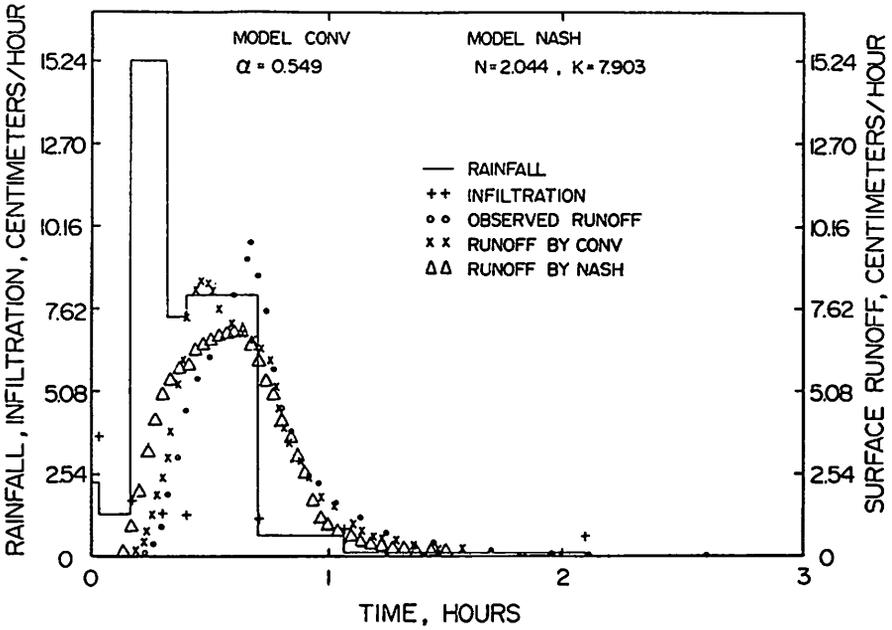


Fig. 2. Hydrograph prediction by kinematic wave model and Nash model for rainfall event of 06/09/1962 on unterraced watershed SW-17, Riesel (Waco), Texas.

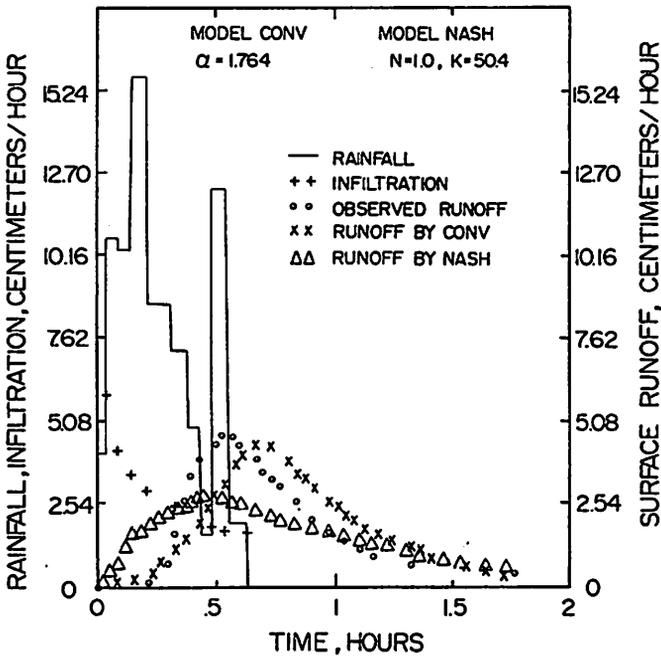


Fig. 3. Hydrograph prediction by kinematic wave model and Nash model for rainfall event of 06/04/1957 on terraced watershed W-2, Riesel (Waco), Texas.

the slopes existing on various portions of a watershed. The parameter, α , considered in the regression analysis, was the optimized value for each watershed. Of all the variables, α was most highly correlated on unterraced watersheds with slope, with a correlation coefficient of 0.584, and on terraced watersheds with area with a correlation coefficient of 0.637. The regression equations are, for unterraced watersheds:

$$\alpha = 11.2581 + 32.676 S^{1/2} + 3.466 \log A - 7.090 \log [L_0(1-r)] \quad (18)$$

with multiple-correlation coefficient of 0.70, mean-squared error of 1.465 and seven degrees of freedom; and for terraced watersheds:

$$\alpha = 0.8114 + 0.534 S^{1/2} + 0.489 \log A - 0.172 \log [L_0(r-1)] \quad (19)$$

with multiple-correlation coefficient of 0.643, mean-squared error of 0.4533 and eight degrees of freedom; where A is area (in hectares); S is weighted slope, and $L_0(1-r)$ is length of flow (in meters). The correlation coefficient is statistically significant. Under the assumption of normality:

$$t = \frac{r}{\left(\frac{1-r^2}{N-2}\right)^{0.5}} \quad (20)$$

satisfies a t -distribution with $N-2$ degrees of freedom (d.f.), where r is correlation coefficient and N number of observations. For unterraced watersheds $t = 2.941$, and for terraced watersheds $t = 2.655$. At 5%-confidence level, 9 d.f., $t_{0.05} = 2.262$, which is less than t for unterraced watersheds. At the same confidence level, 10 d.f., $t_{0.05} = 2.228$, which is less than t for terraced watersheds. One reason that slope is not significantly correlated with α may be that the weighted slope and the parameter $L_0(1-r)$, as obtained from the map, are not good indices of watershed slope and length of flow on terraced watersheds. Terraces are generally constructed with relatively constant slopes and the total length of terraces is probably more closely related to watershed area than to the length parameter used. One way to improve the correlation of α with physiographic parameters of unterraced watersheds would be to include in the analysis several watersheds with a wide range of slopes and lengths. The analysis performed above is only suggestive, and not conclusive. This aspect undoubtedly needs further investigation.

Comparison with Nash model

In the following, NASH will refer to the linear model described by Nash (1957); CONV will refer to the kinematic wave model with a converging geometry. NASH is a linear model which contains two parameters — one denotes the number of linear reservoirs, and the other is a reservoir coefficient. These parameters were estimated for each watershed for the same set of events as for CONV. Based on the converging-section geometry optimization

was performed by the modified Rosenbrock's method (Himmelblau, 1972, pp. 158–167), using the same objective function (eq. 13). Using the optimized parameters, hydrographs were predicted by both NASH and CONV for the same events on each watershed. A sample performance of NASH is shown in Figs. 2 and 3, and Tables III and IV.

A comparison of NASH with CONV would require an answer to the question: what should be the objective criterion for comparing one model with another?

Two aspects must be incorporated in any comparison criterion. One is accuracy, and the other is computational efficiency. The former implies that the hypothesized model should somehow encompass those physical details that affect the dynamics of surface runoff, and hence be able to reproduce it as closely as possible. The latter must include: (1) efficiency of computation; (2) ease of programming; and (3) storage (computer) requirements. One model may be more accurate than the other but may be more costly to operate or require more data. Then, some kind of trade-off may be suggested. Accuracy probably forms a necessary requirement for prediction without prior data. The requirement of efficiency may be met by satisfying the computational efficiency requirement. To compare the two models, the objective function (eq. 17) of CONV was 50.10 and of NASH 76.17 for 48 available events on unterraced watersheds; for 39 available events on terraced watersheds it was 54.5 and 68.76, respectively, for CONV and NASH. This indicates that CONV performed better than NASH, which is also evident from Tables I–IV. On an individual event basis, both models predicted hydrograph peaks, almost equally well, but CONV predicted time characteristics much better than did NASH, also shown by Figs. 2 and 3.

The relative performance of the two models was also compared using the correlation coefficient between runoff observations and predictions, which was 0.95 for CONV and 0.77 for NASH on unterraced watersheds and 0.78 for CONV and 0.77 for NASH on terraced watersheds. This further demonstrates, at least for unterraced watersheds, that CONV is a better predictor.

We noticed that CONV had a tendency to overpredict while NASH had a tendency to underpredict. NASH has a slight advantage over CONV from a computational standpoint but the parameter of CONV could be estimated from watershed physiography.

DISCUSSION AND CONCLUSIONS

Model CONV predicted the hydrograph shape and time characteristics satisfactorily yet was not significantly superior to NASH in predicting hydrograph peaks. One remarkable feature of the model was that it predicted multi-peaked hydrograph characteristics remarkably well in spite of its containing only one parameter, which was optimized using only the hydrograph peak (eq. 17). This suggested that the model embodied an adequate physical structure for describing surface runoff.

TABLE III

Prediction of hydrograph peak characteristic by Nash model on two unterraced watersheds, Riesel (Waco), Texas

Watershed identification	Date of event	Observed hydrograph peak (cm/h)	Predicted hydrograph peak (cm/h)	Error in predicted hydrograph peak	Observed hydrograph peak time (min)	Predicted hydrograph peak time (min)	Error in predicted hydrograph peak time
<i>Watershed Y-4</i> $N = 4.72$ $K = 12.38$	04/24/1957	4.09	3.72	0.090	48.0	70.0	-0.458
	05/13/1957	2.90	2.88	0.006	47.0	66.0	-0.404
	06/04/1957	4.04	3.13	0.225	37.0	61.0	-0.650
	03/29/1965	6.35	5.37	0.153	60.0	130.0	-0.970
<i>Watershed Y-7</i> $N = 0.5$ $K = 51.5$	04/24/1957	6.00	4.36	0.274	31.0	22.0	0.290
	05/13/1957	5.16	3.32	0.356	33.0	26.0	0.212
	06/23/1959	4.47	2.89	0.354	57.0	104.0	-0.825
	03/29/1965	5.17	5.27	0.088	79.0	65.0	0.177

TABLE IV

Prediction of hydrograph peak characteristics by Nash model on two unterraced watersheds near Hastings, Nebraska

Watershed identification	Date of event	Observed hydrograph peak (cm/h)	Predicted hydrograph peak (cm/h)	Error in predicted hydrograph peak	Observed hydrograph peak time (min)	Predicted hydrograph peak time (min)	Error in predicted hydrograph peak time
<i>Watershed W-3</i> <i>N = 4.63</i> <i>K = 9.77</i>	09/05/1946	1.40	1.77	-0.366	146.0	158.0	-0.082
	06/05/1949	0.36	0.58	-0.627	74.0	59.0	0.203
	06/25/1951	1.70	1.55	0.088	129.0	170.0	-0.318
	07/13/1952	3.38	3.51	-0.038	124.0	140.0	-0.129
	06/15/1947	2.49	2.34	0.059	64.0	75.0	-0.172
	06/16/1957	1.73	1.82	-0.052	154.0	162.0	-0.052
<i>Watershed 4-H</i> <i>N = 9.95</i> <i>K = 1.00</i>	08/11/1939	4.52	4.55	-0.007	5.0	12.0	-1.400
	06/20/1942	5.82	5.60	0.037	8.0	15.0	-0.875
	09/05/1946	3.89	3.23	0.169	12.0	12.0	0.000
	06/01/1951	6.16	1.90	0.719	123.0	11.0	0.910
	07/13/1952	9.19	8.93	0.029	21.0	16.0	0.236
	06/12/1958	0.92	1.15	-0.255	14.0	14.0	0.000
	06/12/1965	9.10	8.42	0.132	19.0	19.0	0.000
	06/12/1965	6.14	5.54	0.100	7.0	17.0	-1.429
06/29/1965	8.10	7.30	0.099	3.0	12.0	-3.000	

NASH performed well in predicting hydrograph peak as shown in Figs. 2 and 3 and Tables III and IV. However, an examination of entire hydrographs indicated that time and shape characteristics were not predicted well even though two parameters could be varied in the optimization process.

Because of its nonlinearity CONV was more sensitive to input errors. (Rainfall-excess forms input to the model.) Evidently the observed runoff hydrograph for complex events was not consistent with the computed rainfall-excess for several events. Presumably, the error is in the rainfall-excess. Singh (1974) has a more elaborate discussion on the effect of input error on model results.

From this study we can make the following conclusions.

(1) Model CONV is a highly simplified abstraction of a natural watershed, yet it adequately represents the dynamics of surface runoff with only one fitting parameter.

(2) Geometric parameters of the converging section geometry can be determined from topographic map of the watershed.

(3) A remarkable feature of model CONV is that parameter estimation, based on hydrograph peaks, results in predictions that match the time and shape characteristics of the hydrograph.

(4) Both models CONV and NASH perform well in predicting hydrograph peaks. However, time and shape characteristics are better represented by CONV than NASH.

(5) The parameter, α , is correlated with physical characteristics of a watershed. This aspect, however, needs further exploration.

ACKNOWLEDGEMENT

This research was supported by the Agricultural Research Service, USDA, in cooperation with Colorado State University Experiment Station, Fort Collins, Colorado.

APPENDIX A — ANALYTICAL SOLUTION OF CHARACTERISTIC EQUATIONS

In the context of surface runoff it will be reasonable to partition the problem into: (a) when the lateral inflow q exists; and (b) when q ceases to exist. The first part refers to the rising hydrograph, and the second to the recession hydrograph. Two additional situations must also be considered.

(1) Equilibrium: if the duration of lateral inflow is greater than or equal to the time of equilibrium of the watershed.

(2) Partial equilibrium: if the duration of lateral inflow is less than the time of equilibrium of the watershed.

Partial-equilibrium hydrographs can be derived from equilibrium hydrographs. Therefore, we will present analytic solutions for the equilibrium case only.

Rising hydrograph

Upon dividing eq. 9 by eq. 8 we obtain:

$$nh^{n-1} \frac{dh}{dx} - \frac{(1-r)h^n}{1-(1-r)x} = q \quad (\text{A-1})$$

Eq. A-1 conforms with Bernoulli's differential equation. Using the substitution $W = h^n$, eq. A-1 can be rewritten as:

$$\frac{dW}{dx} - \frac{(1-r)W}{1-(1-r)x} = q \quad (\text{A-2})$$

Eq. A-2 is a first-order first-degree linear nonhomogeneous equation whose solution is:

$$W = \frac{q[1-(1-r)(\frac{1}{2}x)]x}{1-(1-r)x} + \frac{\text{IC}}{1-(1-r)x} \quad (\text{A-3})$$

where IC = constant of integration.

With the initial condition, $h = h_0$ at $x = x_0$ eq. A-3 becomes:

$$h = \left[\frac{h_0^n \{1-(1-r)x_0\}}{1-(1-r)x} - \frac{q \{x_0 - (1-r) \frac{1}{2}x_0^2\}}{1-(1-r)x} + \frac{q \{x - (1-r) \frac{1}{2}x^2\}}{1-(1-r)x} \right]^{1/n} \quad (\text{A-4})$$

Eq. A-4 is a general solution to eq. A-1. Now we solve for eq. 8 of the characteristic ground curve. Substituting eq. A-4 into eq. 8 we obtain:

$$\frac{dx}{dt} = n \left[\frac{h_0^n \{1-(1-r)x_0\} - q \{x_0 - (1-r) \frac{1}{2}x_0^2\} + q \{x - (1-r) \frac{1}{2}x^2\}}{1-(1-r)x} \right]^{\frac{(n-1)}{n}} \quad (\text{A-5})$$

Let:

$$A_1 = h_0^n [1-(1-r)x_0] - q [x_0 - (1-r) \frac{1}{2}x_0^2]$$

$$A_2 = 1 + \frac{2A_1(1-r)}{q}$$

$$A_3 = \frac{1}{n} \left(\frac{1}{1-r} \right)^{\frac{1}{n}} (\frac{1}{2}q)^{\frac{(1-n)}{n}}$$

$$\psi = \frac{[1-(1-r)x]^2}{A_2}$$

We write eq. A-5 as:

$$\frac{dx}{dt} = n \left[\frac{A_1 + q \{x - (1-r) \frac{1}{2} x^2\}}{1 - (1-r)x} \right]^{\frac{(n-1)}{n}} \quad (\text{A-6})$$

Using these transformations, we obtain from eq. A-6:

$$dt = -\frac{A_3}{2} A_2^{\frac{1}{2n}} \psi^{\left(1 - \frac{1}{2n}\right) - 1} (1 - \psi)^{\frac{1}{n} - 1} \quad (\text{A-7})$$

With the initial condition $t = t_0$ at $x = x_0$, the solution to eq. A-7 becomes:

$$(t - t_0) = \frac{A_3}{2} A_2^{\frac{1}{2n}} \left[\beta_a \left(1 - \frac{1}{2n}; \frac{1}{n}\right) - \beta_b \left(1 - \frac{1}{2n}; \frac{1}{n}\right) \right] \quad (\text{A-8})$$

where:

$$a = \frac{[1 - (1-r)x_0]^2}{A_1} \quad \text{and} \quad b = \frac{[1 - (1-r)x]^2}{A_2} \quad (\text{A-9})$$

β with a subscript denotes the incomplete beta-function with the following mathematical connotation:

$$\beta_x(a_1; a_2) = \int_0^x p^{a_1 - 1} (1-p)^{a_2 - 1} dp, \quad a_1 > 0; \quad a_2 > 0; \quad x \in (0, 1)$$

$$= \frac{x^{a_1} (1-x)^{a_2}}{a_1} \left[1 + \sum_{i=0}^{\infty} \frac{\beta(a_1 + 1; i + 1)}{\beta(a_1 + a_2; i + 1)} x^{i+1} \right]$$

However, β without a subscript is understood as the complete beta-function, and can be written in terms of gamma-function as:

$$\beta(a_1; a_2) = \frac{\Gamma(a_1) \Gamma(a_2)}{\Gamma(a_1 + a_2)}$$

Eq. A-8 provides a general analytical solution to eq. 8. Upon considering an initially dry surface, i.e., $h_0 = 0$ at $t_0 = 0$, the solution given by eq. A-8 to eq. 8 simplifies to:

$$t = \frac{A_3}{2} A_2^{\frac{1}{2n}} \frac{\Gamma\left(1 - \frac{1}{2n}\right) \Gamma\left(\frac{1}{n}\right)}{\Gamma\left(1 + \frac{1}{2n}\right)} \frac{\psi^{\left(1 - \frac{1}{2n}\right)} (1 - \psi)^{\frac{1}{n}}}{1 - \frac{1}{2n}} \times$$

$$\times \left[1 + \sum_{j=0}^{\infty} \frac{\beta(2 - \frac{1}{2n}; j+1)}{\beta(1 + \frac{1}{2n}; j+1)} \psi^{j+1} \right] \quad (\text{A-10})$$

Eq. A-10 is a general solution of the characteristic ground curve. The series in eq. A-10 is absolutely convergent, and converges rapidly. If the dimensionless lateral inflow q , is taken as $2r(1+r)$ then, $h = Q = 1$ for steady-state conditions at the downstream boundary.

Recession hydrography

For the recession hydrograph the lateral inflow ceases to exist. Mathematically, the solutions for this case form a special case of the solutions for the rising part but with different boundary conditions.

With $q = 0$, eq. 9 becomes:

$$\frac{dh}{dt} = \frac{(1-r)h}{n[1-(1-r)x]} \quad (\text{A-11})$$

With the initial condition $h = h_0$ at $x = x_0$, the solution to eq. A-11 becomes:

$$h = h_0 \left[\frac{1-(1-r)x_0}{1-(1-r)x} \right]^{\frac{1}{n}} \quad (\text{A-12})$$

For recession from equilibrium, the initial depth h_0 can be calculated from steady-state profile by simple hydraulic considerations, and it will be:

$$h_0 = \left[\frac{x_0 \{2-(1-r)x_0\}}{2\{1-(1-r)x_0\}} \right]^{\frac{1}{n}} \quad (\text{A-13})$$

Thus the discharge at the downstream boundary ($x = 1$) will be given by:

$$Q = \frac{x_0 \{2-(1-r)x_0\}}{1+r} \quad (\text{A-14})$$

The differential equation for the characteristic ground curve becomes:

$$\frac{dx}{dt} = n h_0^{n-1} \left[\frac{1-(1-r)x_0}{1-(1-r)x} \right]^{\frac{(n-1)}{n}} \quad (\text{A-15})$$

Consider the initial condition $t = t_0$ where $x = x_0$, and let t be the time at which the characteristic intersects the lower boundary as shown in Fig. A-1. The general solution to eq. 24 becomes:

$$t = t_0 + \frac{\frac{(n-1)}{2^n}}{(1-r)(2-1)} \left[\frac{\frac{(2n-1)}{\{1-(1-r)x_0\}^n} - \frac{(2n-1)}{\{1-(1-r)x\}^n}}{x_0 \{2-(1-r)x_0\}^n} \right] \quad (A-16)$$

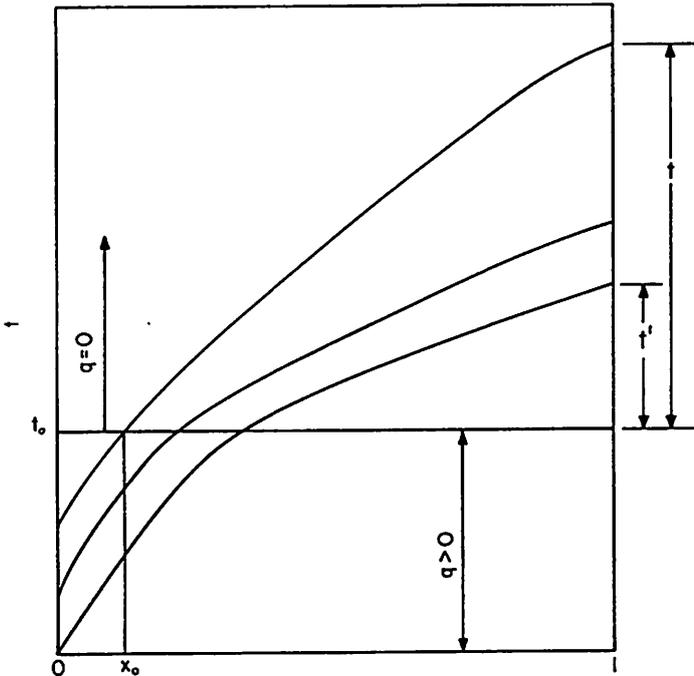


Fig. A-1. Kinematic wave solution domain for converging kinematic flow model.

APPENDIX B — NUMERICAL SOLUTION OF CHARACTERISTIC EQUATIONS

The analytical development in Appendix A assumes that the input (lateral inflow, rainfall) is invariant in time and space. This is a rather severe restriction and greatly limits the application of analytical solutions. To overcome this problem a finite-difference method based on the single-step second-order Lax-Wendroff scheme (Houghton and Kasahara, 1968) was derived. For a definition sketch of the notation see Fig. B-1. The choice of this method is based on the findings of Kibler and Woolhiser (1970) for a plane.

Consider NK nodal points along the x -axis at which the depth of flow is to be computed. Assume the boundary conditions to be:

$$h(x,0) = 0 \quad \text{and} \quad h(0,t) = 0$$

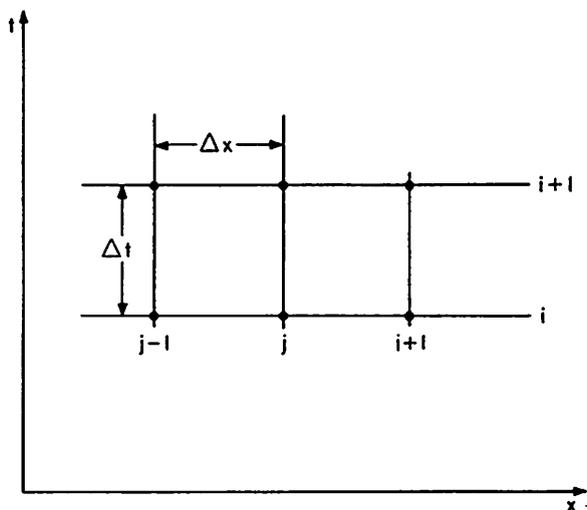


Fig. B-1. Notation for finite-difference scheme.

The finite-difference form of eq. 7, as derived by Singh (1974), can be written as:

$$\begin{aligned}
 h_j^{i+1} = h_j^i & - \left[\frac{(h_{j+1}^i)^n - (h_{j-1}^i)^n}{2\Delta x} - q_j^i - \frac{(1-r)(h_j^i)^n}{1-(1-r)x_j^i} \right] \left[\Delta t + \frac{(\Delta t)^2}{2} \right. \\
 & - \frac{(1-r)n h_j^i}{1-(1-r)x_j^i} \left. \right] + \frac{(\Delta t)^2}{2} \left[\frac{n}{\Delta x} \left(\frac{h_{j+1}^i + h_j^i}{2} \right)^{n-1} \left\{ \left(\frac{(h_{j+1}^i)^n - (h_j^i)^n}{\Delta x} \right) \right. \right. \\
 & - \left. \left. \frac{q_{j+1}^i + q_j^i}{2} - \frac{(1-r) \left(\frac{h_{j+1}^i + h_j^i}{2} \right)^n}{1 - \frac{(1-r)}{2} (x_{j+1}^i + x_j^i)} \right\} - \frac{n}{\Delta x} \left(\frac{h_j^i + h_{j-1}^i}{2} \right)^{n-1} \left\{ \left(\frac{(h_j^i)^n - (h_{j-1}^i)^n}{2} \right) \right. \right. \\
 & \left. \left. - \left(\frac{q_j^i + q_{j-1}^i}{2} \right) - \frac{(1-r) \left(\frac{h_j^i + h_{j-1}^i}{2} \right)^n}{1-(1-r) \left(\frac{x_j^i + x_{j-1}^i}{2} \right)} \right\} + \frac{q_j^{i+1} - q_j^i}{\Delta t} \right] \quad (B-1)
 \end{aligned}$$

and:

$$h_j^{i+1} = h_j^i + \Delta t \left[-n (h_j^i)^{n-1} \left(\frac{h_j^i - h_{j-1}^i}{\Delta x} \right) + q_j^i + \frac{(1-r)(h_j^i)^n}{1-(1-r)x_j^i} \right] \quad (B-2)$$

Eq. B-1 will give, when used in with the boundary conditions, the depth of

flow at nodal points $j = 1, 2, \dots, (NK-1)$; while eq. B-2 will give at the nodal point NK. The use of numerical solutions would require determination of Δx , and Δt (finite-difference increments in space and time) a priori. For the Lax-Wendroff scheme a reasonable number of Δx increments for any watershed is between 10 and 15 (Kibler and Woolhiser, 1970). A Δt increment that will ensure stability of explicit schemes can be found by observing the following stability criterion:

$$\Delta t \leq \min \frac{\Delta x}{C_j}, \quad j = 1, 2, \dots, NK \quad (\text{B-3})$$

where C_j is the kinematic wave celerity at the j^{th} nodal point, and can be computed using eq. 8. The solution to eq. A-1 gives the depth of flow at all points in space for a given point in time. Substituting it in eq. 8 will yield Δt if Δx is specified, which can be done as mentioned above. Determination of Δt in this manner is analytic, and the time-step length (Δt) varies from one step of computation to another. Not only is the stability criterion observed in this way but the resulting Δt would be optimal (i.e., minimum error propagation will take place). In analytical determinations of Δt , the input q is assumed to be constant in time which is consistent because computations are carried out for each pulse in the complex rainfall pattern. For a given pulse, q is constant, even though it varies from pulse to pulse, so we do not violate the condition imposed by the temporal variability of q .

REFERENCES

- Amoroch, J. and Orlob, G.T., 1961. Nonlinear analysis of hydrologic systems. Univ. Calif., Water Resour. Center, Contrib. No. 40, 147 pp.
- Bidwell, V.J., 1971. Regression analysis of nonlinear catchment systems. Water Resour. Res., 7(5): 1118-1126.
- Brakensiek, D.L., 1967a. Kinematic flood routing. Trans. ASAE, 10(3): 340-343.
- Brakensiek, D.L., 1967b. A simulated watershed flow system for hydrograph prediction: A kinematic application. Proc. Int. Hydrol. Symp., Ft. Collins, Colo.
- Chow, V.T. and Ben-Zvi, A., 1973. Hydrodynamic modeling of two-dimensional watershed flow. J. Hydraul. Div., Proc. ASCE, 99(11): 2023-2040.
- Conte, S.D., 1965. Elementary Numerical Analysis: An Algorithm Approach. McGraw-Hill, New York, N.Y., 434 pp.
- Eagleson, P.S., 1972. Dynamics of flood frequency. Water Resour. Res., 8(4): 878-898.
- Harley, B.M., Perkins, F.E. and Eagleson, P.S., 1970. A modular distributed model catchment dynamics. MIT, R.M. Parsons Laboratory for Water Resources and Hydrodynamics, Cambridge, Mass., Rep. 133.
- Himmelblau, D.M., 1972. Applied Nonlinear Programming. McGraw-Hill, New York, N.Y., 416 pp.
- Houghton, D.D. and Kasahara, A., 1968. Nonlinear shallow fluid flow over an isolated ridge. Comm. Pure Appl. Math., 21: 1-23.
- Kibler, D.F. and Woolhiser, D.A., 1970. The kinematic cascade as a hydrologic model. Colo. State Univ. (Fort Collins), Hydrol. Pap., No. 39, 28 pp.
- Lighthill, M.J. and Whitham, G.B., 1955. On kinematic waves, 1. Flood movements in long rivers. Proc. R. Soc. London, Ser. A, 229: 281-316.

- Nash, J.E., 1957. The form of the instantaneous unit hydrograph. IASH Publ., 45(3): 113-121.
- Philip, J.R., 1957. The theory of infiltration, 1. The infiltration equation and its solution. Soil Sci., 84: 257-264.
- Singh, V.P., 1974. A nonlinear kinematic wave model of surface runoff. Ph.D. Dissertation, Colorado State University, Fort Collins, Colo. (unpublished).
- USDA (U.S. Department of Agriculture), 1963. Hydrologic data for experimental watersheds in the United States 1956-1959. U.S. Government Printing Office, Washington, D.C., Misc. Publ. No. 945.
- Wooding, R.A., 1965a. A hydraulic model for the catchment-stream problem, 1. Kinematic wave theory. J. Hydrol., 3(3): 254-267.
- Wooding, R.A., 1965b. A hydraulic model for the catchment-stream problem, 2. Numerical solutions. J. Hydrol., 3(3): 268-282.
- Wooding, R.A., 1966. A hydraulic model for the catchment-stream problem, 3. Comparison with runoff observations. J. Hydrol., 4(1): 21-37.
- Woolhiser, D.A., 1969. Overland flow on a converging surface. Trans. ASAE, 12(4): 460-462.
- Woolhiser, D.A. and Liggett, J.A., 1967. Unsteady, one-dimensional flow over a sloping plane. The rising hydrograph. Water Resour. Res., 3(3): 753-771.