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Overland Flow on a Converging Surface

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MATHEMATICAL watershed models that take into account the physical mechanism of surface runoff consist of various geometric combinations of the basic components of overland and channel flow. In rather simple systems (a parking lot, for example), there may be a one-to-one correspondence between components of the real system and components of the model. If one considers even a small agricultural watershed, however, the number of planes and channels for such a correspondence would become extremely large; so, further simplification is necessary.

In a definitive series of papers, Henderson and Wooding (1)* and Wooding (2), (3), (4) presented analytic and numerical solutions of the kinematic-wave equations for surface runoff from a V-shaped watershed (Fig. 1) and compared numerical results with measurements from three natural watersheds. In Wooding's comparison all watersheds, regardless of their complexity, were represented as a single V-shaped watershed with overland flow planes contributing lateral inflow to a channel in the apex of the V. Although agreement between observed and computed hydrographs was quite good, he concluded that a better geometrical description of the stream network would be desirable. One feature of the hydrographs that his model was unable to reproduce was the steeply rising portion of the hydrograph caused by concentration of runoff.

A watershed model consisting of a V-shaped section plus a portion of the surface of a cone (Fig. 2) at the upstream end may result in a better geometrical description because of the concentration of flow on the cone. Such a model could be taken to represent a watershed of any complexity or it could be used as a basic element in a network model.

Veal (5) derived the continuity and momentum equations for unsteady flow with lateral inflow on a converging surface and obtained numerical solutions for certain values of the parameters. However, his solutions were confined to subcritical flow and he experienced numerical difficulties when the parameters were in the range of hydrologic significance. Woolhiser and Liggett (6) demonstrated that, for one-dimensional overland flow, the kinematic wave approximation is valid for most cases of hydrologic interest, and it appears that it will be sufficiently accurate for converging flow.

This paper presents solutions of unsteady converging flow in dimensionless form using the kinematic wave approximation and examines some of the properties of these solutions.

Basic Equations

The basic equations describing converging overland flow as derived by Veal (5) are:

The continuity equation,

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = q + \frac{uh}{(L_o - x)} \quad \ldots \ldots \ldots \ldots \ldots [1]$$

and the momentum equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = g\,(S_o - S_f) - \frac{q}{h}(u - v) \quad \ldots \ldots \ldots \ldots [2]$$

where u is the local average velocity, h is the local depth, q is the rate of lateral inflow and has dimensions of volume per unit area per unit time, S_o is the bed slope, S_f is the friction slope, g is the acceleration of gravity, v is the x component of the velocity of the lateral inflow, L_o is the radius of the flow region, and x and t are space and time coordinates (Fig. 2).

In the kinematic wave approximation, all terms in the momentum equation, except the term involving bed slope and friction slope, are assumed to be negligible. Equation [2] then becomes

$$S_f = S_o \quad \ldots \ldots \ldots \ldots \ldots [3]$$

which can be written in the form

$$u = \alpha h^{n-1} \quad \ldots \ldots \ldots \ldots [4]$$

Using the Chezy friction formula α would be $C\sqrt{S_o}$ and n would be 3/2.

Equations [1] and [4] can be written in dimensionless form by introducing the following variables:

$$x_* = \frac{x}{L_o(1-r)};\; h_* = \frac{h}{H_o};\; t_* = \frac{V_o t}{L_o(1-r)};\; q_* = q/q_o$$

where $L_o(1-r)$ is the length of the flow plane, H_o is the normal depth at $x = L_o(1-r)$ under steady-state conditions, V_o is the steady-state normal velocity at $x = L_o(1-r)$, and q_o is a normalizing lateral discharge to be chosen later.

By substituting these quantities into equations [1] and [4] we obtain

$$\frac{\partial h_*}{\partial t_*} + u_*\frac{\partial h_*}{\partial x_*} + h_*\frac{\partial u_*}{\partial x_*} = \frac{q_o}{H_o}\frac{L_o}{V_o}(1-r)\,q_* + \frac{(1-r)\,u_*\,h_*}{[1-x_*(1-r)]} \quad \ldots [5]$$

and

$$u_* = h_*^{n-1} \quad \ldots \ldots \ldots \ldots [6]$$

If q_o is chosen to be $\frac{H_o V_o}{L_o(1-r)}$ and q_o and u_* are substituted into [5] we obtain (dropping the asterisks)

$$\frac{\partial h}{\partial t} + nh^{n-1}\frac{\partial h}{\partial x} = q + \frac{(1-r)\,h^n}{[1-x(1-r)]} \quad \ldots \ldots \ldots \ldots [7]$$

which is a partial differential equation having a single family of characteristics.

The differential equation of the characteristic ground curves is:

$$\frac{dx}{dt} = nh^{n-1} \quad \ldots \ldots \ldots \ldots [8]$$

and the time derivative along this curve is:

$$\frac{dh}{dt} = q + \frac{(1-r)\,h^n}{[1-(1-r)x]} \quad \ldots [9]$$

which is a nonlinear, nonhomogeneous ordinary differential equation of the first order and degree.

I have been unable to obtain a general analytical solution to equations [8] and [9], but they are readily solved by finite difference techniques.

When lateral inflow ceases, equation [9] becomes

$$\frac{dh}{dt} = \frac{(1-r)\,h^n}{[1-(1-r)x]} \quad \ldots \ldots [10]$$

If equation [8] is inverted and multiplied by equation [10], there results:

$$\frac{dh}{dx} = \frac{(1-r)\,h}{n[1-(1-r)x]} \quad \ldots \ldots [11]$$

which has the solution:

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* Numbers in parentheses refer to the appended references.

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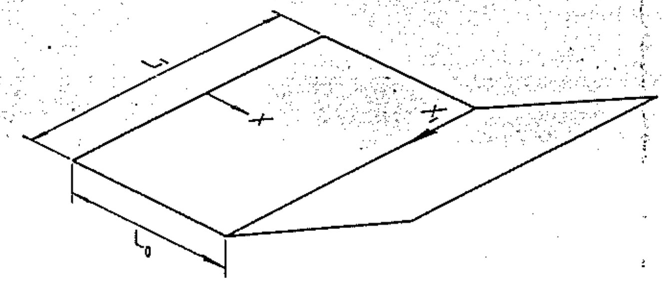

FIG. 1 Geometry of Woodings' runoff model.

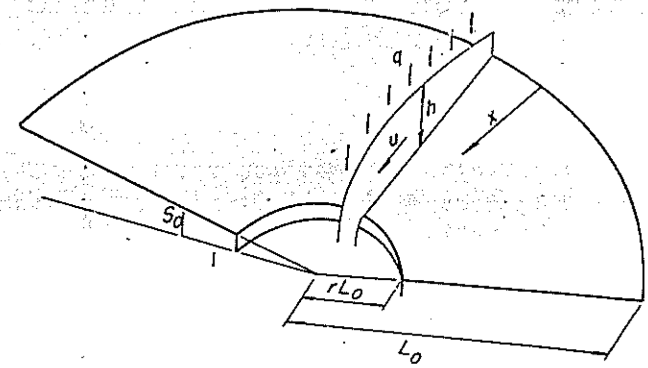

FIG. 2 Geometry of converging section.

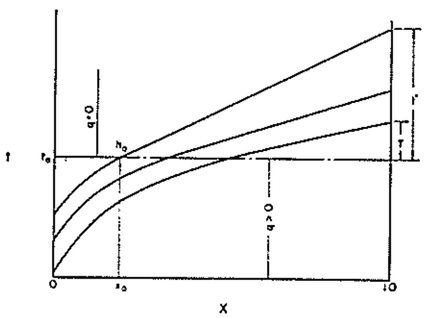

FIG. 3 Solution domain.

$$lnh = -\frac{1}{n} ln\ [1-(1-r)x] + ln\ C_1 \qquad [12]$$

With the following initial conditions

at $x = x_o$, $h = h_o$,

equation [12] becomes

$$h = \frac{[1-(1-r)x_o]^{1/n} h_o}{[1-(1-r)x]^{1/n}} \qquad [13]$$

If the expression for h given by equation [13] is substituted into equation [8], the resulting equation can be solved for t' the time after the lateral discharge stops until a characteristic beginning at x_o, t_o intersects x (Fig. 3 and equation [14]).

The Rising Hydrograph

Rising hydrographs computed by numerical integration along the characteristics with boundary conditions $h(o, t) = 0$; $h(x, o) = 0$, are shown in Fig. 4. The exponent n in equation [6] was 3/2 for this run and the dimensionless lateral inflow rate was

$$q_* = \frac{2r\ (1-r)}{(1-r^2)} = \frac{2r}{1+r}$$

The degree of convergence of flow is indicated by the parameter r. A small value of r indicates a high degree of convergence. As r approaches one, the flow approaches that of flow on a plane surface.

Recession Hydrograph

Two cases of the recession hydrograph are considered herein: (*a*) recession from equilibrium and (*b*) recession from partial equilibrium.

The recession from equilibrium can be obtained analytically by substituting the expression for h_o at equilibrium, as shown in equation [15], into equation [13] and into equation [14] with $x = 1$.

Although the two equations resulting from this substitution could be combined into one equation expressing t' as a function of discharge Q, the resulting expression is quite unwieldy, so Q and t' are written as functions of the parameter x_o (see equation [16] and [17]).

Recessions computed according to equations [16] and [17] are shown in Fig. 4. The lateral inflow rate was set equal to zero at $t = 1.8$ for each case.

Rises and recessions from partial equilibrium are shown in Fig. 5. These solutions were obtained numerically and in all cases the lateral inflow rate was set equal to zero at $t = t_o$ when

$$t' = \left\{\frac{h_o[1-(1-r)x_o]^{1/n}}{(2n-1)(1-r)}\right\}^{-(n-1)} \left\{[1-(1-r)x_o]^{\frac{2n-1}{n}} - [1-(1-r)x]^{\frac{2n-1}{n}}\right\} \qquad [14]$$

$$h_o = \left\{\frac{r}{(1+r)} \frac{x_o\ [2-(1-r)\ x_o]}{[1-(1-r)\ x_o]}\right\}^{1/n} \qquad [15]$$

$$Q = \frac{x_o\ [2-(1-r)\ x_o]}{(1+r)} \qquad [16]$$

$$t' = \left(\frac{1+r}{r}\right)^{\frac{n-1}{n}} \frac{\left\{[1-(1-r)\ x_o]^{\frac{2n-1}{n}} - (r)^{\frac{2n-1}{n}}\right\}}{(2n-1)(1-r)\left\{x_o\ [2-(1-r)\ x_o]\right\}^{\frac{n-1}{n}}} \qquad [17]$$

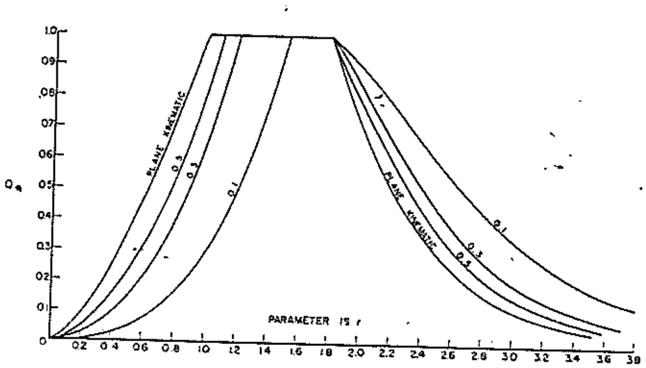

FIG. 4 Rising hydrograph and recession from equilibrium.

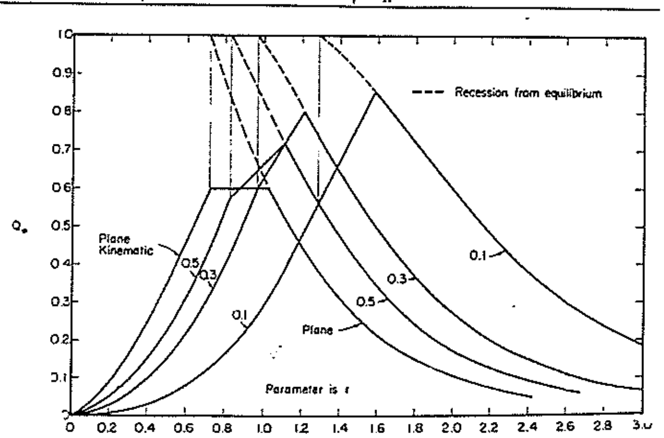

FIG. 5 Rise and recession from partial equilibrium.

$Q \approx 0.6$. Just as in the plane kinematic case, the partial recession curves coincide with recession from equilibrium after an interval T beginning at $t = t_o$. Within the interval $t_o < t < t_o + T$ the hydrograph for the plane kinematic case is flat-topped, whereas the hydrographs for values of $r < 1$ continue to rise (apparently linearly) until they intersect the recession curve. It appears that when data show a rise in the hydrograph after lateral inflow has stopped, it is an indication of dynamic behavior or convergence, or both. Convergence of flow can occur on plane runoff plots with erodible surfaces because of the coalescence of small rivulets.

Conclusions

The characteristic equations for kinematic flow on a converging surface

$$\frac{dx}{dt} = nh^{n-1}$$

and

$$\frac{dh}{dt} = q + \frac{(1-r)h^n}{[1-(1-r)x]}$$

with boundary conditions $h(o, t) = 0$; $h(x, o) = 0$

have been solved numerically for the rising hydrograph and for the recession from partial equilibrium for a number of values of the convergence parameter r. An analytic solution is given for recession from equilibrium.

An examination of the response of a section of a cone to step and pulse inputs of lateral inflow shows that the shape of the rising and falling hydrographs may be changed appreciably by varying the parameter r. A better fit to steeply rising hydrographs may be obtained by including a converging component in a mathematical watershed model. The utility of such a component must be demonstrated by an analysis of experimental or field data.

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