A SEDIMENT YIELD EQUATION FROM AN EROSION SIMULATION MODEL[1]

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ABSTRACT

Sediment is widely recognized as a significant pollutant affecting water quality. To assess the impact of land use and management practices upon sediment yield from upland areas, it is necessary to predict erosion and sediment yield as functions of runoff, soil characteristics such as erodibility, and watershed characteristics. The combined runoff-erosion process on upland areas was modeled as overland flow on a plane, with rill and interrill erosion. Solutions to the model were previously obtained for sediment concentration in overland flow, and the combined runoff-erosion model was tested using observed runoff and sediment data. In this paper, the equations are integrated to produce a relationship between volume of runoff and total sediment yield for a given storm. The sediment yield equation is linear in runoff volume, but nonlinear in distance and, thus, watershed area. Parameters of the sediment yield equation include the hydraulic resistance parameter, rill and interrill erodibility terms, and flow depth-detachment coefficient and exponent.

INTRODUCTION

Sediment is recognized as a significant water pollutant. In addition, sediment transports chemicals, which compounds its effect on water quality. Therefore, it is important to assess the impact of various land use and management practices upon sediment yield from upland areas. To do this, it is necessary to predict erosion and sediment yield as functions of runoff and soil erodibility.

Overland flow on a plane, as a function of time and space, can be modeled by the kinematic wave equations. These equations were developed for flow on hydraulically smooth planes, but have been shown to also apply to irregular surfaces (Woolhiser, Hanson, and Kuhlman, 1970). Thus, for geometrically simple watersheds (topographically uniform) such as upland areas without extensive channel systems, the kinematic wave equations for a plane may be used to compute the overland flow hydrograph.

As a first approximation, erosion on upland areas can be conceptualized as consisting of rill and interrill erosion. Interrill erosion is assumed due to rainfall impact and associated transport in overland flow. Rill erosion is assumed due to tractive forces and subsequent transport capacity in flow occurring in rills or small channels. These rills or small channels are the irregularities in the overland flow surface. The interrill process is the principal source of soil detachment, and also provides transport of soil particles to the rills where they can be transported in the rill flow. Rill flow is the principal source of transport for soil detached from the interrill areas, but it can also be the source of erosion or a site for sediment deposition if the available sediment load exceeds the rill transport capacity. These concepts are summarized mathematically by Equations 7 and 8.

In this paper we derive a simplified sediment yield equation incorporating these concepts. The equation is applied to data from a small watershed, and predicted values are compared with similar values from fitting the Universal Soil Loss Equation (USLE) to determine if the predicted sediment yields are consistent with observations and predictions using a well-known technique.

MODEL

Kinematic wave equations for overland flow on a plane have been shown to apply to many irregular surfaces (eg., Woolhiser, Hanson, and Kuhlman, (1970)). Such surfaces include topographically simple upland areas on natural watersheds. The one-dimensional kinematic wave equations for a plane are:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = R, \tag{1}$$

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and

$$q = Kh^m, \tag{2}$$

where h, q, and R are, respectively, depth of flow, runoff rate, and rainfall excess per unit width of the plane. The functions h, q, and R depend on time, t, and distance, x, down the plane. Also, K is a slope-resistance coefficient, and m is a dimensionless exponent.

These equations are subject to the boundary conditions

$$q(t, 0) = 0 \text{ for } t \geq 0, \tag{3}$$

and

$$q(0, x) = 0 \text{ for } x \geq 0. \tag{4}$$

Erosion accompanying overland flow on topographically simple surfaces can be modeled using equations for rill and interrill erosion on a plane (Hjelmfelt, Piest, and Saxton, 1975; Foster, Meyer, and Onstad, 1973; and others). For flow as described above, and with sediment particles traveling with the mean water velocity, the erosion equations are:

$$\frac{\partial(ch)}{\partial t} + \frac{\partial q_s}{\partial x} = E_I + E_R, \tag{5}$$

where

$$q_s = cq, \tag{6}$$

$$E_I = K_I R, \tag{7}$$

and

$$E_R = K_R(Bh^n - q_s), \tag{8}$$

and where c, q_s, E_I, and E_R are, respectively, sediment concentration, sediment discharge rate, interrill erosion rate, and rill erosion rate per unit width of the plane. Finally, K_I is an interrill coefficient, K_R and B are rill coefficients, and n is a dimensionless exponent.

The erosion equations are subject to the restriction:

c(t, x) is uniformly bounded for all t, $x \geq 0.$ \hfill (9)

In the following development, we assume m = n so that using Equation 2, Equation 8 can be rewritten as:

$$E_R = K_R(\frac{B}{K} q - q_s). \tag{10}$$

MATHEMATICAL CONSEQUENCES OF THE MODEL

We derive a relation between total runoff and sediment yield. To this end we define

$$Q(x) = \text{total runoff} = \int_0^\infty q(t, x) \, dt, \tag{11}$$

$$V(x) = \text{total rainfall excess per unit area} = \int_0^\infty R(t, x) \, dt, \tag{12}$$

and

$$Q_s(x) = \text{total sediment yield} = \int_0^\infty q_s(t, x) \, dt. \tag{13}$$

Notice that for an individual storm, R = 0 after some time $t_\star$, and hence V could have been written as the integral of R from time 0 to $t_\star$. Since Equations 1 - 2 result in a runoff rate, q, which approaches 0 as time tends to ∞, but is never 0 except as specified in the boundary conditions, the infinite integral in Equation 11 is essential. For convenience, we use 0 and ∞ as limits on all time integrals.

We need the following facts which follow from Equations 3, 4, and 6:

$$Q(0) = \int_0^\infty q(t, 0) \, dt = \int_0^\infty 0 \, dt = 0, \tag{14}$$

and

$$Q_s(0) = \int_0^\infty q_s(t, 0) \, dt = \int_0^\infty c(t, 0) \, q(t, 0) \, dt = \int_0^\infty 0 \, dt = 0. \tag{15}$$

Total runoff at a point down the plane should equal total rainfall excess to that point. We demonstrate this mathematically. Integrating Equation 1 with respect to time produces:

$$\int_0^\infty \frac{\partial h(t, x)}{\partial t} \, dt + \int_0^\infty \frac{\partial q(t, x)}{\partial x} \, dt = \int_0^\infty R(t, x) \, dt = V(x) \tag{16}$$

Since initial and final depths of flow are zero, we have

$$\int_0^\infty \frac{\partial h(t, x)}{\partial t} \, dt = h(\infty, x) - h(0, x) = 0. \tag{17}$$

Assuming that we can interchange the integral and derivative,

$$\int_0^\infty \frac{\partial q(t, x)}{\partial x} \, dt = \frac{\partial}{\partial x} \int_0^\infty q(t, x) \, dt = \frac{\partial Q(x)}{\partial x} = Q'(x) \tag{18}$$

Equations 16 - 18 give $Q'(x) = V(x)$, and, hence,

$$Q(x) = \int_0^x V(x) \, dx. \tag{19}$$

If rainfall excess is uniform over the plane, then R depends only on t, and hence V is a constant. Thus,

$$Q(x) = xV, \text{ for uniform rainfall excess.} \tag{20}$$

The relation between Q and Q_S is not directly evident from physical considerations. We derive it from the erosion equations by integrating Equation 5 with respect to time:

$$\int_0^\infty \frac{\partial(ch(t, x))}{\partial t} \, dt + \int_0^\infty \frac{\partial q_s(t, x)}{\partial x} \, dt = \int_0^\infty E_I(t, x) \, dt + \int_0^\infty E_R(t, x) \, dt \tag{21}$$

Since initial and final depth of flow are 0 and concentration is bounded,

$$\int_0^\infty \frac{\partial(ch(t, x))}{\partial t} \, dt = ch(\infty, x) - ch(0, x) = 0. \tag{22}$$

Assuming that we can interchange the integral and derivative,

$$\int_0^\infty \frac{\partial q_s(t, x)}{\partial x} \, dt = \frac{\partial}{\partial x} \int_0^\infty q_s(t, x) \, dt = \frac{\partial Q_s(x)}{\partial x} = Q'_s(x) \tag{23}$$

Integrating Equation 7 and since $Q'(x) = V(x)$

$$\int_0^\infty E_I(t, x) \, dt = K_I \int_0^\infty R(t, x) \, dt = K_I V(x) = K_I Q'(x). \tag{24}$$

Integrating Equation 10 gives

$$\int_0^\infty E_R(t, x) \, dt = K_R \left(\frac{B}{K} \int_0^\infty q(t, x) \, dt - \int_0^\infty q_s(t, x) \, dt \right) = K_R \left(\frac{B}{K} Q(x) - Q_s(x) \right) \tag{25}$$

Equations 21 - 25 combine to give $Q'_s = K_I Q' + K_R (\frac{B}{K} Q - Q_s)$.

This can be rewritten as $L' + K_R L = (K_I - \frac{B}{K})Q' = (K_I - \frac{B}{K})V$, where $L = Q_s - \frac{B}{K}Q$. Solving for L gives

$$L(x) = e^{-K_R x} \int_0^x e^{K_R s} (K_I - \frac{B}{K})V(s) \, ds + \text{const}.$$

Since $L(0) = 0$ by Equations 14 -15, we have

$$Q_s(x) = \frac{B}{K}Q(x) + e^{-K_R x} \int_0^x e^{K_R s} (K_I - \frac{B}{K})V((s) \, ds. \tag{26}$$

If rainfall excess is uniform on the plane, then V is constant and Equation 26 can be integrated to give

$$Q_s(x) = \frac{B}{K}Q(x) + (K_I - \frac{B}{K})V(\frac{1-e^{-K_Rx}}{K_R}).$$

Combining with Equation 20 gives

$$Q_s(x) = Q(x)\ (\frac{B}{K} + (K_I - \frac{B}{K})(\frac{1-e^{-K_Rx}}{K_Rx})), \tag{27}$$

which is a simplified sediment yield equation for uniform rainfall excess. The above derivation also works when the topographical model is uniform across the width of the surface, but has varying slope. In this case K becomes a function of x, reflecting the variation in slope. It may also be desirable in this more general model to let B, K_I, and K_R also vary with x. The basic relation $Q'_s = K_IQ' + K_R(\frac{B}{K}Q - Q_s)$ still holds. However, Equations 26 and 27 require that these parameters be uniform in space.

PARAMETER ESTIMATION

The erosion model, Equations 5 to 9, has four parameters, K_I, K_R, B, and n, and the runoff model, Equations 1 to 4, has two parameters, K and m. In addition, a number of parameters are required to describe rainfall excess. Foster and Meyer (1971) suggested a value of 1.5 for n. The exponent m is also 1.5 if the Chezy formula is used for turbulent flow. This justifies using m = n in the model. In a particular numerical procedure we would not hesitate to use m = n = 1.5.

Parameter estimation based upon solutions to the coupled runoff and sediment concentration equations. If the partial differential equations (Eqs. 1 to 4 and Eqs. 5 to 9) are solved for runoff rate, q(t, x), and sediment concentration, c(t, x), then optimal parameters can be determined by fitting simulated results to observed data. This was done using a least squares procedure (Lane and Shirley, 1978) for data from a small watershed in Arizona.

Initial estimates of parameters for optimization. Values of the hydraulic resistance parameter, K, can be estimated from tabular values obtained from experimental data (e.g., Lane, Woolhiser, and Yevjevich, 1975), then K_I, K_R, and B can be estimated using measured values of initial, mean, and final concentration. Initial concentration, C_0, can be estimated by extending observed concentration data back to the t = 0 axis on a plot of concentration versus time. Mean concentration, $\overline{c}$, can be estimated as $\overline{c} = Q_s/Q$, given values of sediment yield and runoff volume. Finally, the terminal or final concentration, c_∞, can be estimated by extending concentration data forward in time through the recession until the end of the event on a plot of concentration versus time.

Given these estimates of, c_0, $\overline{c}$, and c_∞, the equations to solve simultaneously for the parameters K_I, K_R, and B are derived from the model:

$$c_0 = K_I, \tag{28}$$

$$\overline{c} = \frac{B}{K} + (K_I - \frac{B}{K})(1 - e^{-K_Rx})/(K_Rx), \text{ and} \tag{29}$$

$$c_\infty = \frac{B}{K} + (K_I - \frac{B}{K})e^{-K_Rx} \tag{30}$$

The equation for mean concentration was developed above (eq. 27), and Equations 28 and 30 were developed by Lane and Shirley (1978) for the case of constant and uniform rainfall excess. However, they also hold for the more general case of time varying rainfall excess uniform over the plane.

Solution of Equations 28 - 30 provide initial estimates of K_I, K_R, and B to use in an optimization procedure where simulated concentration data are fitted to observed concentration data.

Results. A small (1.3 ha) watershed called Watershed 63.101 on the Walnut Gulch Experimental Watershed (Renard, 1970) was selected for analysis. The 1.3 ha watershed was modeled as a plane of length 194 m, width of 67 m, and total relief of 7.8 m. This watershed is instrumented with rainfall, runoff, and sediment sampling equipment. During periods of runoff following rainstorms, pump-type (suspended sediment) samples are taken at 3-minute intervals.

From 1973 through 1975, eight single peaked hydrographs were selected for analysis. Also, a ninth event with a secondary peak was selected because it was the largest event of record, and it provided an extreme case. For these nine events, optimal parameters (K, K_I, K_R, and B) were determined as described above. Sediment yields were computed (using the optimal parameters) for each event using Equation 27, and compared with observed sediment yield. The equation relating computed sediment yield, Q_s, and observed sediment yield, Q_s, in kg is:

$$\hat{Q}_S = 8.2 + 0.89 \ Q_S \tag{31}$$

with the coefficient of determination, $R^2 = 0.99$. As a comparison, by using the "best" single value of mean concentration (Eq. 29) and observed runoff volumes, Equation 27 was used to predict sediment yields for the nine events. The equation relating computed and measured sediment yields for this case is:

$$\hat{Q}_S = 54.7 + 0.90 \ Q_S \tag{32}$$

with $R^2 = 0.98$.

The Universal Soil Loss Equation (USLE) described by Wischmeier and Smith (1965) is:

$$A = RKLSCP \tag{33}$$

where

    A = estimated soil loss (tons/acre/year),
    R = rainfall factor,
    K = soil erodibility factor,
    L = slope length factor,
    S = slope gradient factor,
    C = cropping factor, and
    P = erosion control practice factor.

The USLE (with KLSCP = 0.0027) was fitted to data from this watershed, and then used to compute sediment yields for the nine events on Watershed 63.101. The least squares equation relating computed and observed sediment yields is:

$$\hat{Q}_S = 24.5 + 0.96 \ Q_S \tag{34}$$

with $R^2 = 0.98$.

Results of computed sediment yields using the USLE (Eq. 34) and the sediment yield equation derived herein (Eq. 32) are quite comparable (Fig. 1). Since the intercepts in both regression equations (Eqs. 32 and 34) are positive, they would both tend to overpredict sediment yields for the very small events. The slope terms in both equations are less than one; therefore, they would both tend to underpredict for the very large events. However, these differences are slight for the range of data analyzed (Fig. 1).

The derived sediment yield equation produces sediment yield computations consistent with observations and with computations using the USLE. Additional testing will be required to determine if these results hold under a variety of watershed conditions.

The KLSCP terms in the USLE are represented by

$$Q_s/Q = \frac{B}{K} + (K_I - \frac{B}{K})(\frac{1-e^{-K_R x}}{K_R x}) \tag{35}$$

in the derived sediment yield equation. The major difference in these equations is that the KLSCP terms represent uniform sediment production over the watershed area while $(\frac{1-e^{-K_R x}}{K_R x})$ in the derived sediment yield equation represents decreasing sediment yield per unit area as the watershed area increases with increasing x. Also, the derived equation models the erosion by component processes and includes runoff volume in the prediction equation. This may be a significant factor in cases where the R term in the USLE is not a good predictor for runoff volume.

BRIEF SENSITIVITY ANALYSIS

We conducted a brief sensitivity analysis to estimate the influence of changes in the parameter values on the computed sediment yields. Each of the four parameters (K, K_I, K_R, and B) was varied over a range of $\pm 60\%$ from its specified value in Equation 27 (as shown in Fig. 1). The resulting percentage change in parameter values vs the percentage change in computed sediment yield is shown in Fig. 2. Notice that computed sediment yields are linear with K_I and B, and nonlinear with K_R and K. Also, these computations are for parameter values around the specified parameter values. Therefore, the relative sensitivity of computed sediment yield to each parameter might be different for other parameter values.

SUMMARY AND CONCLUSIONS

A sediment yield equation is derived from the partial differential equations for overland flow with rill and interrill erosion on a plane. The derived sediment yield equation incorporates hydraulic resistance, rill and interrill erodibility terms, distance (watershed area), and runoff volume. This sediment yield equation, comparable in simplicity to the USLE, was used to compute sediment yields for a number
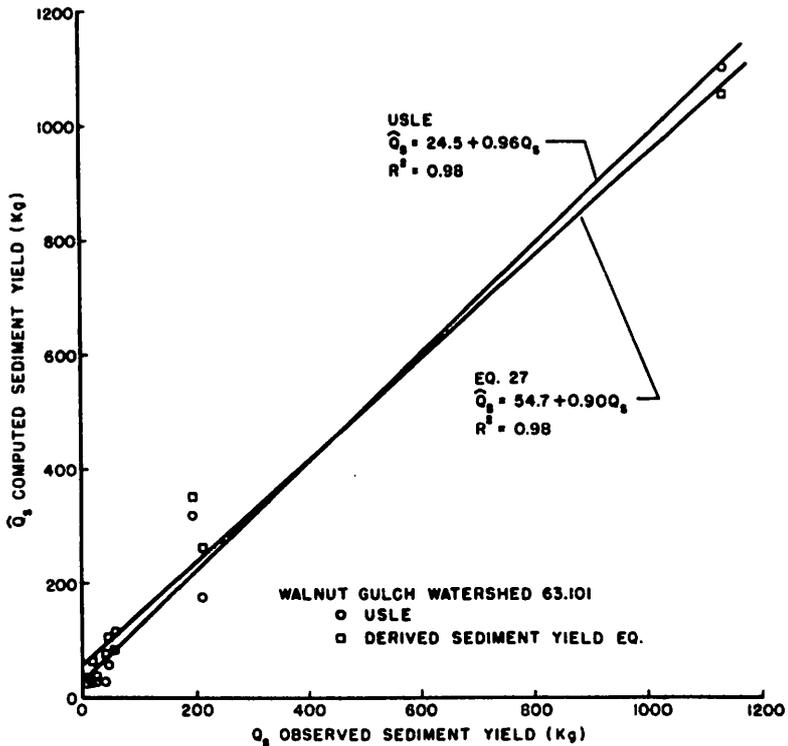
Fig. 1. Relation between observed and computed sediment yield for the
USLE and the derived sediment yield equation. Watershed 63.101.

of events on a small semiarid watershed. Computed sediment yields compared favorably with observations and with estimates made using the USLE. However, the derived sediment yield equation accounts for decreasing sediment yield with increasing watershed area.

Based on our analysis of the sediment yield equation and its properties, we conclude that it produces reasonable estimates for sediment yields on small semiarid watersheds such as Watershed 63.101.

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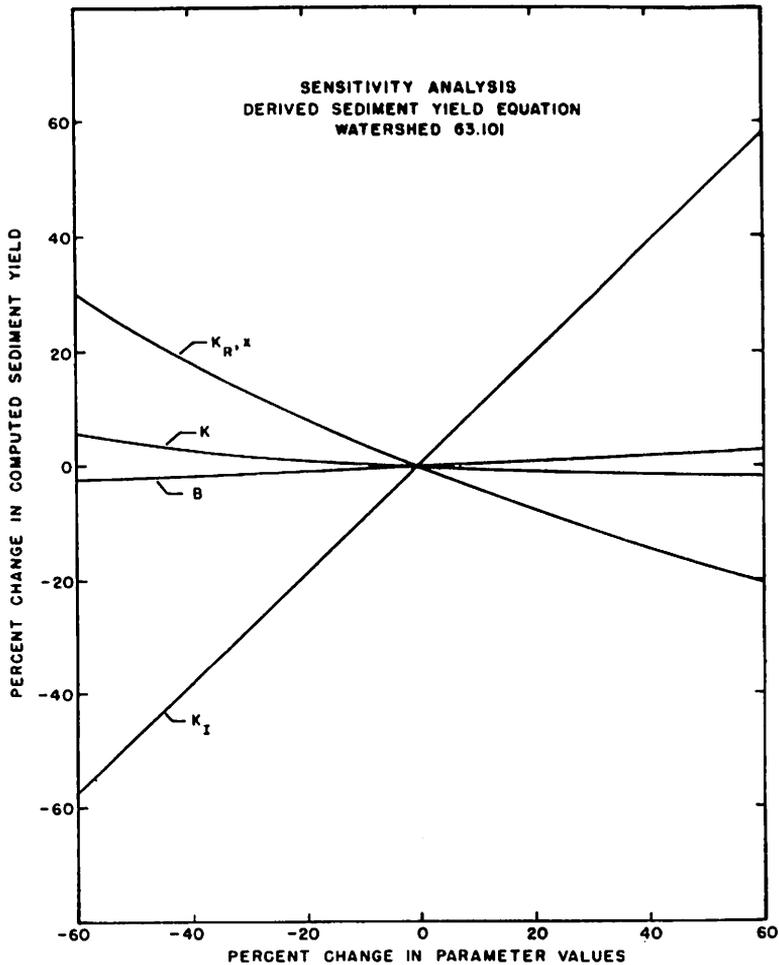
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