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United States-Japan Bi-Lateral Seminar on Hydrology
Honolulu, January 1971

SIMULATION OF THE SHORT-TIME SCOUR-FILL PROCESS IN ERODIBLE STREAMS WITH STOCHASTIC SEDIMENT TRANSFER AT THE STREAM BED

by

J. Paul Riley[1], Kousoum Sakhan[2], and Kenneth G. Renard[3]

SYNOPSIS:

In ephemeral streams, the occurrence of translatory waves causes the shortening of the time of rise of the hydrograph which in turn results in channel instability. A simulation model is developed to describe the dynamics of the channel in terms of: (1) two one-dimensional stream flow equations, (2) a one-dimensional sediment transport equation, an equation for the stream bed, and (3) a stochastic sediment transfer at the stream bed which also includes the bed load.

The model as a whole is simulated on a hybrid computer. To demonstrate the operation of the model, real-time simulation is done using hypothetical data for a stream reach 24,000 feet in length. The results of this study are presented in graphical form.

1. INTRODUCTION

Existing information is inadequate on channel stability influences in the Southwest part of the United States. Under present conditions, many stream channels are unstable. The major cause is believed to be the high-intensity, short-duration convective thunderstorms, particularly during the summer season, which result in flash floods moving over coarse-textured alluvial stream beds with very high-intake rates. It has been observed by Renard and

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Hickok[4] that the occurrence of translatory waves is frequent. This phenomenon has a critical influence on the time of rise of the hydrograph, which in turn, affects the stability of the stream channel. The conventional theory of flood routing has been found to be inapplicable to these ephemeral streams. Therefore, it is necessary to account for the movement of these waves in order for the model to be descriptive of ephemeral streams. Besides these translatory waves, sediemtns carried by the flow both as bed load and suspended load make the situation even more complicated.

It is the purpose of this investigation to develop a simulation model which accounts for the translatory waves, and sediments entrained by the flow both as bed load and suspended load, and yet is simple enough for practical uses. The system reported herein consists of two one-dimensional stream flow equations, a solid mass transport equation, and stochastic solid mass transfer at the stream bed.

2. CHANNEL DYNAMICS MODEL

Fluid Mass and Solid Mass Transport Equations

The three dimensional equations of conservation of fluid mass, flow momentum, and solid mass in the turbulent flow are

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

$$\rho \frac{dV}{dt} = \rho\vec{g} - \nabla p + \mu \nabla^2 \vec{V} + [\nabla \cdot T^{(e)}] \tag{2}$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\vec{V}) = \nabla \cdot (e\nabla c) \tag{3}$$

subject to the following boundary conditions (Figure 1)

$$\frac{\partial F}{\partial t} + \vec{V} \cdot \nabla F = \vec{R} \cdot \nabla F \tag{4}$$

$$c \frac{\partial F}{\partial t} - e \nabla c \cdot \nabla F = \vec{N} \cdot \nabla F + c \vec{R} \cdot \nabla F \tag{5}$$

where $T^{(e)}$ is the Reynolds stress tensor and $F(x, y, z, t) = 0$ is the equation of the moving boundary surface. After space averaging, Equations (1), (2) and (3) reduce to the following one-dimensional equations which describe the conservation of fluid mass, flow momentum, and solid mass in the flow of natural streams with movable boundaries,

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} - L - S = 0 \tag{6}$$

[4] Renard, Kenneth G., and Hickok, R. B., "Sedimentation Research Needs in Semi-Arid Regions," Proc. of the ASCE, Vol. 93, No. HY1, pp. 45-60, January, 1967.

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} (\beta \frac{Q^2}{A}) + g \frac{\partial}{\partial x} (h_c A) + gA \frac{\partial \eta}{\partial x} + \frac{Q^2 F}{8 R_h A} - \frac{Q}{A} (L+S) = 0 \qquad (7)$$

$$\frac{\partial}{\partial t} (CA) + \frac{\partial}{\partial x} (CQ) - \frac{\partial}{\partial x} (G_{cx} A \frac{\partial C}{\partial x}) - C/_{y=\eta} [S + T \frac{\partial \eta}{\partial t}] - N = 0 \qquad (8)$$

where the terms L, S, and N, as described in the terminology, have negative values if processes they represnet are influent, and positive values if processes are effluent. The scour-fill equation for the stream bed is

$$C/_{y=\eta} \; T \frac{\partial \eta}{\partial t} + \frac{\partial Q_{B\ vol}}{\partial x} + N = 0 \qquad (9)$$

In natural streams, different bed configuration occurs depending upon the flow characteristics. Therefore, the friction factor, f, which appears in Equation (7) is by no means invariant. This friction factor can be decomposed into f' which represents the sand grain roughness and f'' which represents the bed form roughness. These two friction factors are represented by the following equations

$$f' = 1/ [2 \log_{10} (2 R_h/D_{50}) + 1.74]^2 \qquad (10)$$

$$\log_{10} f'' = - .45 - .04 \frac{U}{\sqrt{gD_{50}}} + (.04 \frac{U}{\sqrt{gD_{50}}} - 3.05)$$

$$\exp - 8 [\log_{10} \frac{R_b}{D_{50}} - (1.9 + .04 \frac{U}{\sqrt{gD_{50}}})] \qquad (11)$$

In the equation for the suspended sediment (Equation (8)), two variables must be determined: (1) the longitudinal sediment dispersivity G_{cx}, and (2) the sediment transfer rate N at the stream bed.

Following Chen[5] and assuming logarithmic velocity distribution in both vertical and lateral directions, the longitudinal sediment dispersivity can be derived and expressed as:

$$G_{cx} = \frac{\lambda k^2}{24} \frac{\sqrt{h}}{R_h} \frac{[\log_e (\frac{h}{\eta_0 - \eta_1}) - 1]}{\log_e \left(\frac{T/2}{\xi_0 \; \xi_1}\right)} T^2 \qquad (12)$$

in which λ is the ratio of the eddy mass diffusivity to the eddy kinematic viscosity of water, k is the von Karman universal constant, h is the depth of flow, T is

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[5] Chen, C. L., "Dispersion of Sediment in Flow with Moving Boundaries," accepted for publication in the Proc. of the ASCE, Hydraulics Division, 1971.

the top width of flow, and η_o and ξ_o are the distances from the stream bed and the side walls, respectively, where the velocity is zero.

The rate of transfer of sediment N should be described by a stochastic process as a result of the random nature of the driving mechanism which is the flow turbulence. This process is treated in the following section.

Stochastic Sediment Transfer at the Stream Bed

It is proposed that at the interface between the flow region and the stream bed, there are three distinct states and at time, t, a solid particle may be in any one of these three states. The three states are: A the suspended load state, B the bed load state, and C the immobile bed state (Figure 2). The probability that a particle will move from one state to another or remain in a particular state is termed its transition probability. The future transition probabilities of a solid particle are independent of its transition probabilities in the past (a property of the Markov processes). For example, if a particle is in state A at time θ and in state B at time ξ the probability that is in state C at time t is given by the appropriate transition probabilities as follows:

$$P_{AC}(\theta,t) = P_{AB}(\theta,\xi)\, P_{BC}(\xi,t) \tag{13}$$

Generally, since the events corresponding to Equation (13) for different middle states are mutually exclusive, then the probability of going from state A at time θ to state C at time t is

$$P_{AC}(\theta,t) = \sum_{i=A}^{C} P_{Ai}(\theta,\xi)\, P_{iC}(\xi,t) \tag{14}$$

Equation (14) is the Chapman-Kolmogorov equation for a non-homogeneous Markovian process. If it is defined that

$$\psi_{AB}\,\Delta t + 0\,(\Delta t) = P_r\ \{\text{a particle in state A at time } \xi \text{ will be in state B at time } \xi + \Delta t\} \tag{15}$$

$$\psi_{AC}\,\Delta t + 0\,(\Delta t) = P_r\ \{\text{a particle in state A at time } \xi \text{ will be in state C at time } \xi + \Delta t\} \tag{16}$$

and

$$1 + \psi_{AA}\,\Delta t + 0\,(\Delta t) = P_r\ \{\text{a particle in state A at time } \xi \text{ will remain in state A during the interval } (\xi, \xi + \Delta t)\} \tag{17}$$

then it is clear that

$$\psi_{AA} = - \sum_{i=B}^{C} \psi_{Ai} \leq 0 \tag{18}$$

If ψ_{AA} is zero, then state (A) is absorbing, that is, the particle is always in suspension. This case is not expected to happen for dense solid particles wandering in a natural stream.

Now consider two contiguous time intervals, (θ, t) and $(t, t + \Delta t)$. Using the definitions in Equations (15) through (17) with Equation (14) yields

$$P_{AA}(\theta, t + \Delta t) = P_{AA}(\theta,t)(1 + \psi_{AA}\Delta t) + P_{AB}(\theta,t)\psi_{BA}\Delta t + P_{AC}(\theta,t)\psi_{CA}\Delta t + 0(\Delta t) \tag{19}$$

$$P_{AB}(\theta, t + \Delta t) = P_{AA}(\theta,t)\psi_{AB}\Delta t + P_{AB}(\theta,t)(1 + \psi_{BB}\Delta t) + P_{AC}(\theta,t)\psi_{CA}\Delta t + 0(\Delta t) \tag{20}$$

and

$$P_{AC}(\theta, t + \Delta t) = P_{AA}(\theta,t)\psi_{AC}\Delta t + P_{AB}(\theta,t)\psi_{BC}\Delta t + P_{AC}(\theta,t)(1 + \psi_{CC}\Delta t) + 0(\Delta t) \tag{21}$$

Dividing Equation (19) through (21) by Δt and taking the limit as Δt approaches zero results in the following relationships:

$$\frac{\partial}{\partial t} P_{AA}(\theta,t) = P_{AA}(\theta,t)\psi_{AA}(t) + P_{AB}(\theta,t)\psi_{BA}(t) + P_{AC}(\theta,t)\psi_{CA}(t) \tag{22}$$

$$\frac{\partial}{\partial t} P_{AB}(\theta,t) = P_{AA}(\theta,t)\psi_{AB}(t) + P_{AB}(\theta,t)\psi_{BB}(t) + P_{AC}(\theta,t)\psi_{CB}(t) \tag{23}$$

$$\frac{\partial}{\partial t} P_{AC}(\theta,t) = P_{AA}(\theta,t)\psi_{AC}(t) + P_{AB}(\theta,t)\psi_{BC}(t) + P_{AC}(\theta,t)\psi_{CC}(t) \tag{24}$$

Equations (22) through (24) are known as the forward Kolmogorov differential equations. They are subject to the following initial conditions

$$P_{Ai}(\theta,\theta) = \delta_{Ai}, \; i = A, B, C \tag{25}$$

where δ_{Ai} is the Kronecker delta symbol.

The ψ's are the intensities of solid mass transfer which are the main keys to the concept of stochastic solid mass transfer at the interface. They are functions of the instantaneous local hydrodynamic properties. Considering that the time interval (θ, t) is very short, then it can be assumed that the ψ's are constant over this interval. However, in general the ψ's are time-dependent.

The Kolmogorov differential equations for P_{BA}, P_{BB}, P_{BC}, P_{CA}, P_{CB}, and P_{CC} may be derived in a similar manner and presented in matrix format. Let the transition probability matrix be

$$\vec{P}(\theta,t) = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} P_{AA}(\theta,t) & P_{AB}(\theta,t) & P_{AC}(\theta,t) \\ P_{BA}(\theta,t) & P_{BB}(\theta,t) & P_{BC}(\theta,t) \\ P_{CA}(\theta,t) & P_{CB}(\theta,t) & P_{CC}(\theta,t) \end{pmatrix} \end{array} \tag{26}$$

and let the solid mass transfer intensity matrix be

$$\vec{\psi}(t) = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} \psi_{AA}(t) & \psi_{AB}(t) & \psi_{AC}(t) \\ \psi_{BA}(t) & \psi_{BB}(t) & \psi_{BC}(t) \\ \psi_{CA}(t) & \psi_{CB}(t) & \psi_{CC}(t) \end{pmatrix} \end{array} \tag{27}$$

Then the Chapman-Kolmogorov equation becomes

$$\vec{P}(\theta,t) = \vec{P}(\theta,\xi)\,\vec{P}(\xi,t) \tag{28}$$

In a similar manner the forward Kolmogorov differential equations become

$$\frac{\partial}{\partial t}\vec{P}(\theta,t) = \vec{P}(\theta,t)\,\vec{\psi}(t) \tag{29}$$

subject to the initial conditions

$$\vec{P}(\theta,t) = \vec{I} \tag{30}$$

where $\vec{I}$ is the identity matrix.

The task, now, is to determine the intensity matrix $\vec{\psi}(t)$. Referring to Figure 2, there are six from-state-to-state transfers and three within-state transfers. The terms state and zone should not be confused. With the complexity regarding the arrangement of solid particles at the interface and the instantaneous local hydrodynamic forces, it is impossible to draw a physical boundary between the bed load zone and the suspended load zone. However, it is possible to avoid this problem by considering states of a solid particle which do not have physical boundaries, but rather "process boundaries". The process boundaries are defined by the following relations (31) through (36) which define the process conditions for transfers between states. For example, relations (31) and (36) define the process boundaries for state B.

AB - Transfer from suspended load state to bed load state occurs when the local upward hydrodynamic force F_y (v'_{up}) is less than the vertical resisting force F_y (m) due to solid mass.

$$F_y\ (v'_{up}) \leq F_y\ (m) \tag{31}$$

AC - Transfer from suspended load state directly to immobile bed state occurs when

$$F_y\ (v'_{up}) \leq F_y\ (m) \tag{32}$$

BA - Transfer from bed load state to suspended load state takes place when

$$F_y\ (v'_{up}) > F_y\ (m) \tag{33}$$

BC - Transfer from bed load state to immobile bed state occurs when

$$F_x\ (\tau_0,\ u') \leq F_x\ (m,\ \tau_0) \tag{34}$$

CA - Transfer from immobile bed state directly to suspended load takes place when

$$F_y\ (v'_{up}) > F_y\ (m) \tag{35}$$

CB - Transfer from immobile bed state to bed load state occurs when

$$F_x\ (\tau_0,\ u') > F_x\ (m,\ \tau_0) \tag{36}$$

As can be seen from Equation (18), within-state transfer are functions of transfers between states.

Let $u_{forward}$ be the forward velocity of the particle at the interface regions, and v_{rise} and v_{fall}, the rise and fall velocities of the particle, respectively. If a sediment particle moved with one of these three velocities, how many fictitious sediment particles similar to that real particle would move at maximum intensity, past a single "serving counter" per unit time? Let D be the diameter of the particle. Then there should be, for example, $u_{forward}/D$ particles per unit time which move in the forward horizontal direction. Under this concept, the intensity function matrix (27) becomes

$$\vec{\Phi}(t) = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} -\left(\frac{v_{fall}}{D} + \frac{v_{fall}}{D}\right) & \frac{v_{fall}}{D} & \frac{v_{fall}}{D} \\ \frac{v_{rise}}{D} & -\left(\frac{v_{rise}}{D} + \frac{v_{fall}}{D}\right) & \frac{v_{fall}}{D} \\ \frac{v_{rise}}{D} & \frac{u_{forward}}{D} & -\left(\frac{v_{rise}}{D} + \frac{u_{forward}}{D}\right) \end{bmatrix} \end{array} \tag{37}$$

subject to the process conditions (31) through (36). The entry which does not satisfy its respective process condition vanishes.

In order to evaluate the above matrix the three solid particle velocities, $u_{forward}$, v_{rise}, and v_{fall}, must be determined. In natural streams, various bed configurations are formed, depending upon the flow conditions, the characteristics of the bed material, and the stream geometry. In turn, bed features affect "wall" turbulence near the bed, which is a primary entrainment mechanism for solid particles. Another important entrainment mechanism is the mean bed shear stress. In the interface region of natural streams, the Newtonian law of motion is valid. Thus

$$\vec{F} = \frac{d}{dt}(m\,\vec{V}_p) \tag{38}$$

in which $\vec{F}$ is the total force vector on the particle; m is the mass of the particle, which is constant; and $\vec{V}_p$ is the velocity vector of the particle. Integrating Equation (38) from time t_1 to time t_2 yields

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\,\vec{V}_{p_2} - M\,\vec{V}_{p_1} \tag{39}$$

where $\int_{t_1}^{t_2} \vec{F}\,dt$ is known as the impluse of the force $\vec{F}$ on the particle. Decomposing the force $\vec{F}$ into two parts, the hydrodynamic force, $\vec{F}_{hydro}$, and the resisting force, $\vec{F}_{resist}$, and integrating Equation (39) over the entire surface, Λ, of the bed feature yields

$$\int_\Lambda \int_{t_1}^{t_2} \vec{F}_{hydro}\, dt\, d\Lambda + \int_\Lambda \int_{t_1}^{t_2} \vec{F}_{resist}\, dt\, d\Lambda = m \int_\Lambda (\vec{V}_{p_2} - \vec{V}_{p_1})\, d\Lambda \qquad (40)$$

Let $\overline{\vec{F}}_{hydro}$, $\overline{\vec{F}}_{resist}$, $\overline{(\vec{V}_{p2} - \vec{V}_{p1})}$ and $\overline{\Lambda}$ be the averages of $\vec{F}_{hydro}$, $\vec{F}_{resist}$, $(\vec{V}_{p2} - \vec{V}_{p1})$, and Λ and t in the domain, respectively. Then Equation (40) becomes

$$\overline{\Lambda}\,(t_2 - t_1)\,(\overline{\vec{F}}_{hydro} + \overline{\vec{F}}_{resist}) = \overline{\Lambda} m\, \overline{(\vec{V}_{p_2} - \vec{V}_{p_1})} \qquad (41)$$

In natural streams, the longitudinal and vertical motions of bed particles are usually the predominant processes. Hence, from Equation (41),

$$u_p = \overline{(u_{p_2} - u_{p_1})} = \frac{t_2 - t_1}{m}\,(\overline{F}_{hydro\ x} + \overline{F}_{resist\ x}) \qquad (42)$$

and

$$v_p = \overline{(v_{p_2} - v_{p_1})} = \frac{t_2 - t_1}{m}\,(\overline{F}_{hydro\ y} + \overline{F}_{resist\ y}) \qquad (43)$$

where u_p and v_p are the relative longitudinal and vertical particle velocities. The time of action $(t_2 - t_1)$ of the summation of forces $(\overline{F}_{hydro} + \overline{F}_{resist})$ is assumed, as can be deduced from the reasoning to obtain the intensity function matrix (37), to be:

$$t_2 - t_1 = \frac{D}{\pm\,(\vec{V}_{p_2} - \vec{V}_{p_1})} \qquad (44)$$

Because the time of action $(t_2 - t_1)$ should always be positive, the plus and minus signs in front of the expression $(V_{p2} - V_{p1})$ in Equation (44) are necessary.

Substituting Equation (44) into Equations (42) and (43) yields

$$\pm\, u_p^2 = \frac{D}{m}\,(\overline{F}_{hydro\ x} + \overline{F}_{resist\ x}) \qquad (45)$$

and

$$\pm\, v_p^2 = \frac{D}{m}\,(\overline{F}_{hydro\ y} + \overline{F}_{resist\ y}) \qquad (46)$$

Considering the fact that solid particles are sheared over a gravity bed in order to move in the longitudinal direction, one would expect that the particle to particle interaction is far greater in the longitudinal direction than in the vertical direction. Then the longitudinal resisting force for a spherical particle can be expressed as

$$\bar{F}_{resist\ x} = -(\gamma_s - \gamma)\frac{\pi}{6}D^3 \tan\alpha \tag{47}$$

in which $\tan\alpha$ is the dynamic friction factor, which as a result of Bagnold's[6] experiments can be expressed as

$$\tan\alpha = .375 + .375 \exp\left[-.00084\left(\frac{\gamma_s D^2 \tau_0}{14\, g\mu^2} - 100\right)\right] \tag{48}$$

Since the particle to paricle interaction is negligible in the vertical direction, the vertical resisting force is due only to the submerged weight of the particle, or

$$\bar{F}_{resist\ y} = -(\gamma_s - \gamma)\frac{\pi}{6}D^3 \tag{49}$$

The total hydrodynamic force is the mean bed shear stress, $-\rho\overline{u'v'}$, superimposed by forces resulting from turbulent fluctuations u' and v'. Thus,

$$\bar{F}_{hydro\ x} = \frac{\pi}{4}D^2\, 1/2\, C_{Dx}\frac{\gamma}{g}(\overline{u'^2} - \overline{u'v'}) \tag{50}$$

and

$$\bar{F}_{hydro\ y} = \frac{\pi}{4}D^2\,(1/2\, C_{Dy}\frac{\gamma}{g}\overline{v'^2_{up}}) \tag{51}$$

in which C_{Dx} and C_{Dy} are the drag coefficients in the longitudinal and vertical directions, respectively, and $\overline{v'^2_{up}}$ is the mean-square of the upward velocity fluctuation. If asymmetry of turbulent fluctuations is assumed to exist at the stream bed, then, for a maximum local momentum flux, the upward fluctuation can be derived and expressed as:

$$\overline{v'^2_{up}} = 2.415\,\overline{v'^2} \tag{52}$$

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[6] Bagnold, R. A., "Flow of Cohesionless Grains in Fluid", Royal Society [London] Philos. Trans., Vol. 249, pp. 235-297, 1956.

Now let ϕ_x and ϕ_y be the correlation functions between the root-mean-square velocity fluctuations $\sqrt{\overline{u'^2}}$ and $\sqrt{\overline{v'^2}}$ and the root-mean-square shear velocity $\sqrt{-\overline{u'v'}}$, or

$$\phi_x = \sqrt{-\frac{\overline{u'^2}}{\overline{u'v'}}} \qquad (53.a)$$

and

$$\phi_y = \sqrt{-\frac{\overline{v'^2}}{\overline{u'v'}}} \qquad (53.b)$$

Substituting Equations (47) through (53) into Equations (45) and (46) yields

$$\pm\, u_p^{\,2} = 3/4\ g\ \frac{C_{Dx}}{\gamma_s}\,(\phi_x^2 + 1)\ \tau_0 - g\ D\ (\frac{\gamma_s - \gamma}{\gamma_s})\ \tan\alpha \qquad (54)$$

and

$$\pm\, v_p^{\,2} = 1.81\ g\ \frac{C_{Dy}}{\gamma_s}\ \phi_y^2\ \tau_0 - g\ D\ (\frac{\gamma_s - \gamma}{\gamma}) \qquad (55)$$

in which τ_0 is the mean bed shear stress $(-\rho\overline{u'v'})$.

The correlation functions ϕ_x and ϕ_y are functions of the friction Reynolds number (Laufer[7]) or

$$\phi_x = 2.5 - 2.5\ \exp\,(-\ .322\sqrt{\tau_0/\rho}\ D/\nu) \qquad (56.a)$$

and

$$\phi_y = 1.0 - \exp\,(-\ .0525\sqrt{\tau_0/\rho}\ D/\nu) \qquad (56.b)$$

Equations (54) and (55) provide estimates of the three particle velocities required for the intensity function matrix (37). The Kolmogorov differential equation (29) then can be integrated to obtain the probability for each process.

Once the probability for each process is obtained, the concentration, the rate of transfer, and the volume rate of flow of the sediment at the stream bed

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[7] Laufer, J., "The Structure of Turbulence in Fully Developed Pipe Flow," National Advisory Committee for Aeronautics, Technical Report 1174, p. 17, 1954.

are given, respectively, by:

$$C_B(t) = C_A\ (t-\delta t)\ P_{AB}\ (t-1/2\delta t) + C_C\ (t-\delta t)\ P_{CB}\ (t-1/2\delta t)$$
$$- C_B\ (t-\delta t)\ [P_{BA}\ (t-1/2\delta t) + P_{BC}\ (t-1/2\delta t)] \tag{57}$$

$$N(t) = \frac{\partial}{\partial x}\Big\{ BD_{50}\ C_C\ (t-\delta t)\ [\psi_{CB}\ (t-1/2\delta t)\ P_{CB}\ (t-1/2\delta t)$$
$$+ \psi_{CA}\ (t-1/2\delta t)\ P_{CA}(t-1/2\delta t)] - C_A\ (t-\delta t)\ \psi_{AC}(t-1/2\delta t)$$
$$P_{AC}\ (t-1/2\delta t) - C_B\ (t-\delta t)\ \psi_{BC}\ (t-1/2\delta t)\ P_{BC}\ (t-1/2\delta t)\Big\} \tag{58}$$

and

$$Q_{B\ vol}(t) = \Big\{ BD_{50}\ C_C\ (t-\delta t)\ \psi_{CB}\ (t-1/2\delta t)\ P_{CB}\ (t-1/2\delta t)$$
$$+ C_A\ (t-\delta t)\ \psi_{AB}\ (t-1/2\delta t)\ P_{AB}\ (t-1/2\delta t) - \psi_{CB}\ (t-\delta t)$$
$$[\psi_{BA}\ (t-1/2\delta t)\ P_{BA}\ (t-1/2\delta t) + \psi_{BC}\ (t-1/2\delta t)\ P_{BC}\ (t-1/2\delta t)]\Big\} \tag{59}$$

Since the elevation of the stream bed is a function of both the rate of transfer of sediment between the stream bed and the bed load state and the stream bed and the suspended load state, Equation (9) can also be written

$$C/_{y=\eta}\ T\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\ (Q_{B\ vol} + Q_{B\ exc}) = 0 \tag{60}$$

where

$$Q_{B\ exc}(t) = \frac{\partial}{\partial x}\Big\{ BD_{50}\ C_C\ (t-\delta t)\ [\psi_{CB}\ (t-1/2\delta t)\ P_{CB}\ (t-1/2\delta t)$$
$$+\psi_{CA}\ (t-1/2\delta t)\ P_{CA}\ (t-1/2\delta t)] - C_A\ (t-\delta t)\ \psi_{AC}\ (t-1/2\delta t)$$
$$P_{AC}\ (t-1/2\delta t) - C_B\ (t-\delta t)\ \psi_{BC}\ (t-1/2\delta t)\ P_{BC}\ (t-1/2\delta t)\Big\} \tag{61}$$

Completing the formulation, the mathematical model of the scour-fill process in erodible streams can be visualized by the illustration in Figure 3.

The particle diameter, D, used in the development of the stochastic model is for homogeneous bed material. For heterogeneous bed material, the mean particle diameter should be used.

Having arrived at this point, a question still remains. Is the Kolmogorov system (29), which described Markovian processes with time-dependent transition probabilities, a valid system to describe the stochastic sediment transfer at the interface; in other words, is the time spent within each state and from state-to-state exponentially distributed? Exponential distribution is a necessary, though no sufficient, condition of a Markov process, and since the time of particle stay or transfer is a function of the instantaneous particle velocity, this velocity has to possess a distribution which belongs to the exponential family.

To see that the instantaneous particle velocity, $\vec{V}_p$, has an exponential distribution, let $F(\vec{V}_p)$ be its distribution function with a range, Γ, or

$$F(\vec{V}_p) = \int_\Gamma f(\vec{V}_p)\, d\vec{V}_p \tag{62}$$

Going back to Equations (55) and (56) it can be seen that, at the interface,

$$\vec{V}'_1 = +\sqrt{a\,\vec{V}_p^2 + b},\quad \vec{V}'_2 = -\sqrt{a\,\vec{V}_p^2 + b} \tag{63}$$

in which a and b are coefficient scalars. Equation (63) indicates that the density function of the random variable $\vec{V}_p$ is a double-valued function. Therefore, the density function $f(\vec{V}_p)$ can be obtained by partial differentiation of Equation (62) as

$$f(\vec{V}_p) = \frac{\partial F(\vec{V}_p)}{\partial \vec{V}'} \left| \frac{d\vec{V}'_1}{d\vec{V}_p} \right| + \frac{\partial F(\vec{V}_p)}{\partial \vec{V}'_2} \left| \frac{d\vec{V}'_2}{d\vec{V}_p} \right| \tag{64}$$

At the stream bed, the fluctuating velocity $\vec{V}'$ has a nearly Gaussian distribution. Hence,

$$\frac{\partial F(\vec{V}_p)}{\partial \vec{V}'_1} = \frac{1}{\sqrt{2\pi}\ \sigma_{\vec{V}'}} \exp\left[- \frac{(\vec{V}' - \bar{\vec{V}}')^2}{2\,\sigma_{\vec{V}'}} \right] \tag{65}$$

and

$$\frac{\partial F(\vec{V}_p)}{\partial \vec{V}'_2} = \frac{1}{\sqrt{2\pi}\ \sigma_{\vec{V}'}} \exp\left[- \frac{(\vec{V}' + \bar{\vec{V}}')^2}{2\,\sigma^2_{\vec{V}'}} \right] \tag{66}$$

where $\sigma_{\vec{V}'}$ is the standard deviation of $\vec{V}'$ about its mean $\bar{\vec{V}}'$.

Differentiating Equation (63) with respect to $\vec{V}_p$ yields

$$\left| \frac{d\vec{V}'_1}{d\vec{V}_p} \right| = \left| \frac{d\vec{V}'_2}{d\vec{V}_p} \right| = \frac{a\sqrt{c\,\vec{V}'^2 + d}}{\vec{V}'} \tag{67}$$

in which c and d are coefficient scalars. (See Equations (55) and (56)).

Then substituting Equations (65), (66), and (67) into Equation (64) yields

$$f(\vec{V}_p) = \frac{a\sqrt{c\,\vec{V}'^2 + d}}{\vec{V}'\ \sqrt{2\pi}\ \sigma_{\vec{V}'}} \left\{ \exp\left[- \frac{(\vec{V}' - \vec{V}')^2}{2\,\sigma^2_{\vec{V}'}} \right] + \exp\left[- \frac{(\vec{V}' + \vec{V}')^2}{2\,\sigma^2_{\vec{V}'}} \right] \right\} \tag{68}$$

Equation (68) indicates that $f(\vec{V}_p)$ has a Gamma distribution which belongs to the exponential family, and therefore suggests that it might be described by a Markovian process with discrete states in continuous time. If this is the case, the Kolmogorov system (29) is a valid description of stochastic sediment transfer at the interface as was assumed in the development presented herein.

3. METHODS OF SOLUTION

The essence of the model developed in this paper is to obtain the solution to systems of equations: (1) one-dimensional equations based on the conservation of fluid mass, flow momentum, and sediment mass in the flow, and (2) the Kolmogorov differential equations.

To obtain a high-accuracy approximation in the solution of Equations (6) and (7), the one-step Lax-Wendroff is adopted. Equations (6) and (7) are rewritten in the following form:

$$\frac{\partial \vec{W}}{\partial t} + \frac{\partial F(\vec{W})}{\partial x} + \vec{K} = 0 \tag{69}$$

where

$$\vec{W} = \begin{bmatrix} A \\ Q \end{bmatrix}, \quad F(\vec{W}) = \begin{bmatrix} Q \\ \frac{Q^2}{A} + gh_c A \end{bmatrix}, \quad \vec{K} = \begin{bmatrix} -L \; -S \\ gA \frac{\partial n}{\partial x} + \frac{Q^2}{8} \frac{f}{AR_h} \end{bmatrix} \tag{70}$$

The vector $\vec{W}$ can be expressed in Taylor series as follows:

$$\vec{W}_i^{j+1} = \vec{W}_i^j + \Delta t \left(\frac{\partial \vec{W}}{\partial t}\right)_i^j + \frac{\Delta t^2}{2} \left(\frac{\partial^2 \vec{W}}{\partial t^2}\right)_i^j + \ldots \tag{71}$$

Since this is an explicit scheme in time, t, the second order term is preserved in order to obtain high order accuracy. Equation (71) can also be written

$$\vec{W}_i^{j+1} = \vec{W}_i^j - \Delta t \left[\frac{\partial F(\vec{W})}{\partial x} + \vec{K}\right]_i^j + \frac{\Delta t^2}{2}\left[\frac{\partial}{\partial x}\left[\vec{J}_F\left[\frac{\partial F(\vec{W})}{\partial x} + \vec{K}\right]\right]\right]_i^j + \frac{\Delta t^2}{2}\left[\vec{J}_K\left[\frac{\partial F(\vec{W})}{\partial x} + \vec{K}\right]\right]_i^j \tag{72}$$

in which $\vec{J}_F$ and $\vec{J}_K$ are the Jacobians of $F(\vec{W})$ and $\vec{K}$ with respect to $\vec{W}$, respectively, or

$$\mathfrak{J}_F = \begin{bmatrix} 0 & 1 \\ \frac{\partial}{\partial A}(gh_c A) - U^2 & 2U \end{bmatrix} \tag{73}$$

and

$$\mathfrak{J}_K = \begin{bmatrix} -\frac{\partial S}{\partial A} & 0 \\ g\frac{\partial \eta}{\partial x} - \frac{U^2}{8}\frac{f}{R_h} - \frac{Q^2}{8}\frac{f}{AR_h^2}\frac{R_h}{A} + \frac{Q^2}{8\,AR_h}\frac{\partial f}{\partial A} & \frac{Uf}{4\,R_h} + \frac{Q^2}{8\,AR_h}\frac{\partial f}{\partial Q} \end{bmatrix} \tag{74}$$

All space derivatives are approximated by centered differences. Using the von Newmann stability criterion, it can be shown that the following condition must hold for stability in the solution

$$\left[U_{max} + \sqrt{\frac{\partial}{\partial A}(gh_c A)_{max}}\right]\frac{\Delta t}{\Delta x} < 1 \tag{75}$$

Characteristic of the Lax-Wendroff scheme is the presence of short-wave oscillations behind the shock front, particularly when the shock is strong. If these oscillations are allowed to be carried with the computation long enough, they might cause the solution to explode. Therefore, it might be necessary to add, in some cases, a filtering term or artificial viscosity to Equation (72) to minimize these oscillations.

The solution to the suspended sediment Equation (8) can be approximated by the following explicit difference scheme

$$\begin{aligned} C_i^{j+1} A_i^{j+1} &= \frac{C_{i+1}^j A_{i+1}^j + C_{i-1}^j A_{i-1}^j}{2} - \frac{\Delta t}{\Delta x}\left(C_{i+1}^j Q_{i+1}^j - C_i^j Q_i^j\right) \\ &+ \frac{\Delta t}{2(\Delta x)^2}\Big((G_{cx_{i+1}}^{\ j} A_{i+1}^j + G_{cx_i}^{\ j} A_i^j)(C_{i+1}^j - C_i^j) - (G_{cx_i}^{\ j} A_i^j \\ &+ G_{cx_{i-1}}^{\ j} A_{i-1}^j)(C_i^j - C_{i-1}^j)\Big) + \Delta t\, C_{B_i}^{\ j} S_i^j - \frac{\Delta t}{2\Delta x}\left(Q_B\, vol_{i+1}^{\ j+1} - Q_B\, vol_{i-1}^{\ j+1}\right) \end{aligned} \tag{76}$$

In most practical cases, Equation (76) is of positive type; and, therefore, it is always stable.

The Kolmogorov system (29) which consists of nine ordinary differential equations, is solved directly on the analog computer. As far as stability in the solution is concerned, the analog technique is particularly suited for the Kolmogorov system because the intensities of sediment transfer within each state are negative. This causes the errors associated with the computation to decay exponentially with time. The analog diagram for this system is shown in Figure 4.

4. HYBRID COMPUTER EXPERIMENTS

The model as a whole is being synthesized on a hybrid computer available at the Utah Water Research Laboratory. The model will be tested by simulating conditions within an ephemeral reach of the Walnut Gulch Watershed in southern Arizona. This is a highly instrumental watershed operated by the Southwest Watershed Research Center, Agricultural Research Service, Tucson, Arizona. Measured input and control data will be used for calibration and testing of the processes modeled, and the model subsequently will be used for preditive purposes on this watershed.

Since field data are not yet available hypothetical input data are used to demonstrate the operation of the model. The movement of flood wave and sediment is simulated in a reach which is 24,000 feet long. This reach is divided into 20 space intervals, each 1,200 feet long. To stay within the stability limits of the difference equations used, the time increment is taken as one minute. The average slope is about .005.

With the input hydrograph as shown in Figure 5, the wave reaches the downstream end after 35 minutes. It is attenuated to about half the size of the input. This is due mainly to the high intake rate of the stream bed. Figure 5 also shows that the peak of the hydrograph occurs at about the same time as the peak of the suspended sediment graph, while, as expected, the peak of the bed load graph lags behind the hydrograph peak. These phenomena are illustrated by Figures 6, 7, and 8, which show the profiles of the water discharge, bed load discharge, and suspended load discharge, respectively.

A sample of the probability profiles is shown in Figure 9 for the transfer of sediment from immobile bed state to bed load state. The profiles tend to have relatively flat lee-side slopes which are similar to those of the velocity profiles.

5. SUMMARY

A mathematical model of the sediment transport process in ephemeral streams is presented. The main advantage of the approach adopted is that the problem of separating bed load from suspended load is avoided by dealing with states of occurrence rather than physical zones. Stochastic processes are used to describe sediment transfer between states.

The model is being synthesized on a hybrid computer. It was found that the computational schemes used are stable. The term associated with the friction factor in the equation of momentum controls the stability in the solution. The responses of sediment load to change in water discharge are as expected. The suspended load tends to change instantaneously with the water hydrograph, while there is some lag in the response function associated with the bed load.

PARTIAL LIST OF SYMBOLS

Symbol	Definition
A	Cross sectional area of flow
A_C	Cross sectional area of the stream bed
A	Suspended load state at the interface
B	Bed load state at the interface
C	Immobile bed state at the interface
C	Overall average concentration of suspended sediment
C_C	Specific weight of the bed material
C_i	Concentration of sediment at state i at the interface
$C/_{y=\eta} = C_B$	Bed load concentration
c	Point sediment concentration in 3-dimensional field
C_D	Drag coefficient
D	Sediment diameter
D_{50}	Median sediment diameter
e	Mass diffusivity in the 3-dimensional field
$\vec{F}$	Vector force on the particle
$F(x, y, z, t) = 0$	Boundary surface in the 3 dimensional field
f	Friction factor
f'	Friction factor due to grain roughness
f''	Friction factor due to bed forms
G_{cx}	Longitudinal solid dispersivity
$\vec{g}$	Gravitational acceleration
h	Flow depth
h_c	Depth from free stream surface to centroid of flow cross sectional area
I	Intensity of turbulence
$\vec{I}$	Identity vector
L	Lateral flow into or from the channel per unit length

Partial List of Symbols (Continued)

Symbol	Definition
m	Mass of solid particle
N	Local total sediment transfer rate at the stream bed
P_{AB}	Probability that a particle moves from state A to state B
p	Porosity of the bed at static state
p	Point pressure
Q	Total fluid discharge
$\vec{R}$	Vector flow across stream boundary surface
R_h	Hydraulic radius
S	Total seepage into or from the channel bed per unit length
T	Top width of flow
$T^{(e)}$	Reynolds stress tensor
U	Mean stream velocity
$U*$	Friction velocity
$\vec{V}$	Temporal average fluid velocity vector
$\vec{V}'$	Temporal fluid velocity fluctuation vector
$\vec{V}_p$	Solid particle velocity vector at the interface
u', v'	Turbulent velocity fluctuations
α	Average angle of encounter between individual particles
β	Momentum correction factor
γ	Fluid specific weight
γ_s	Solid particle specific weight
δ_{ij}	Kronecker delta symbol
η	Stream bed elevation above some datum
μ	Fluid dynamic viscosity
ν	Fluid kinematic viscosity
ρ	Fluid density
ρ_s	Solid particle density
τ_o	Bed shear stress
$\vec{\psi}$	Vector intensity of sediment motion at the interface

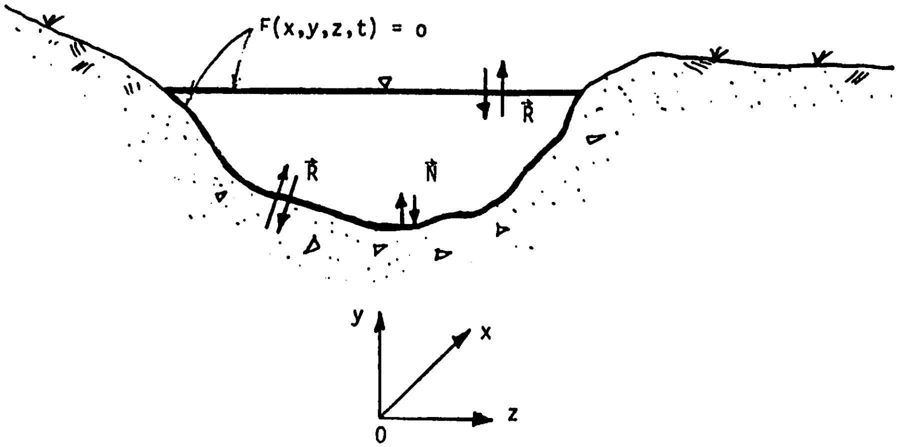


Figure 1. Illustration of a stream cross section.

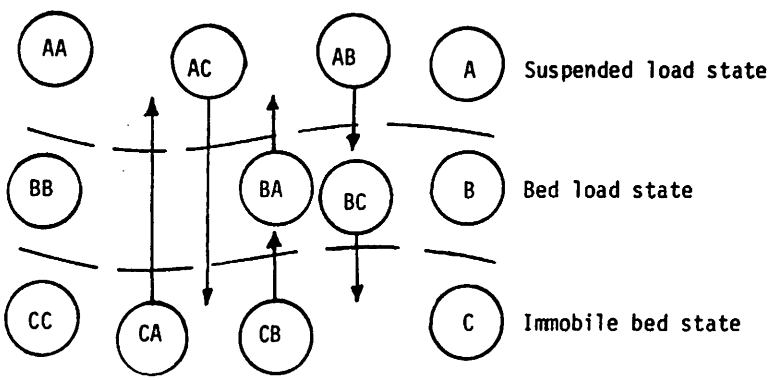


Figure 2. Illustration of solid mass transfer at the interface

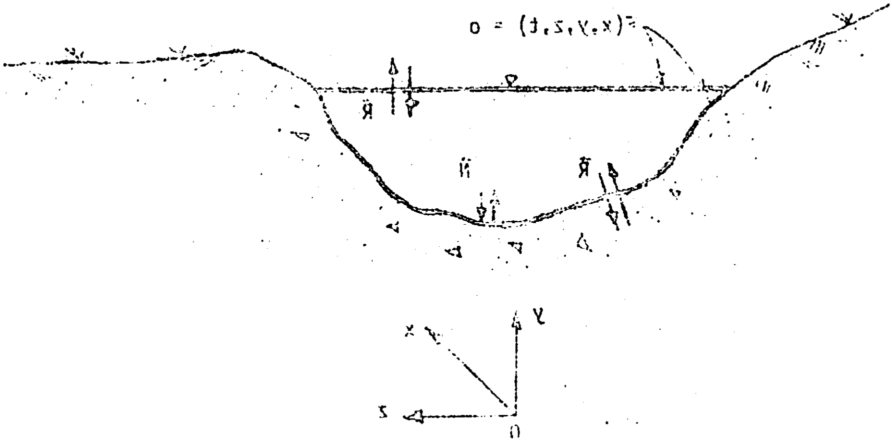


Figure 1. Illustration of a stream cross section.

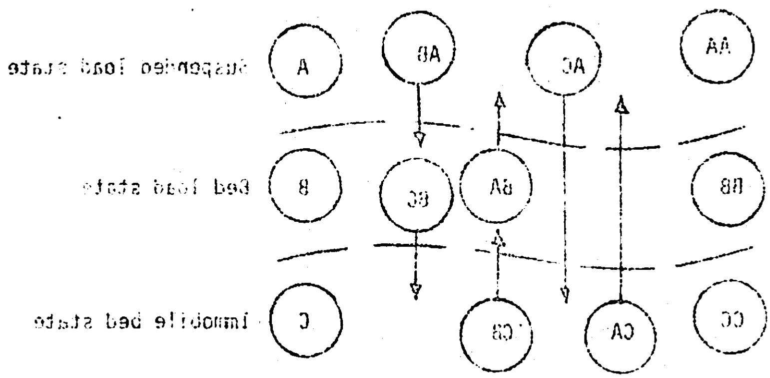


Figure 2. Illustration of solid mass transfer at the interface

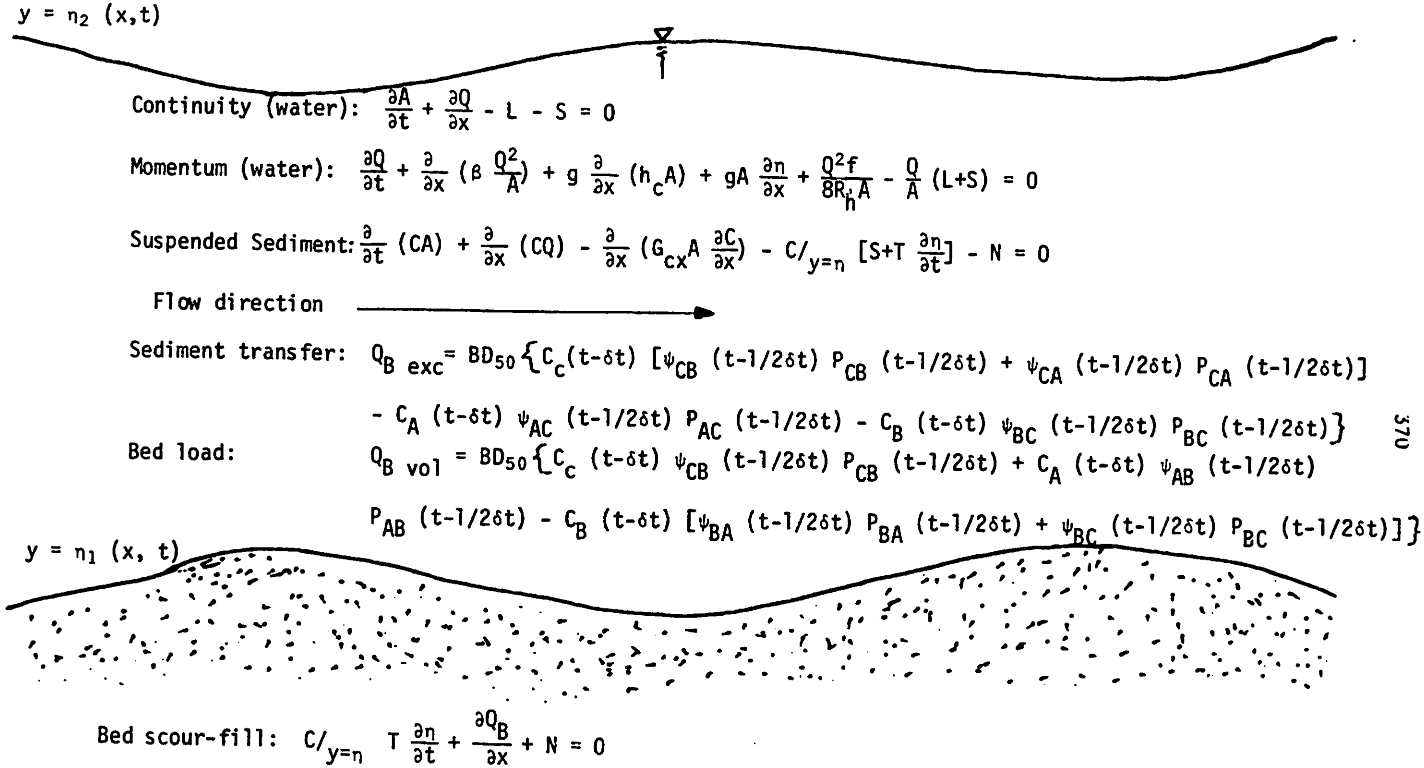


Figure 3. Schematic diagram showing the various equations used in the sediment movement model.

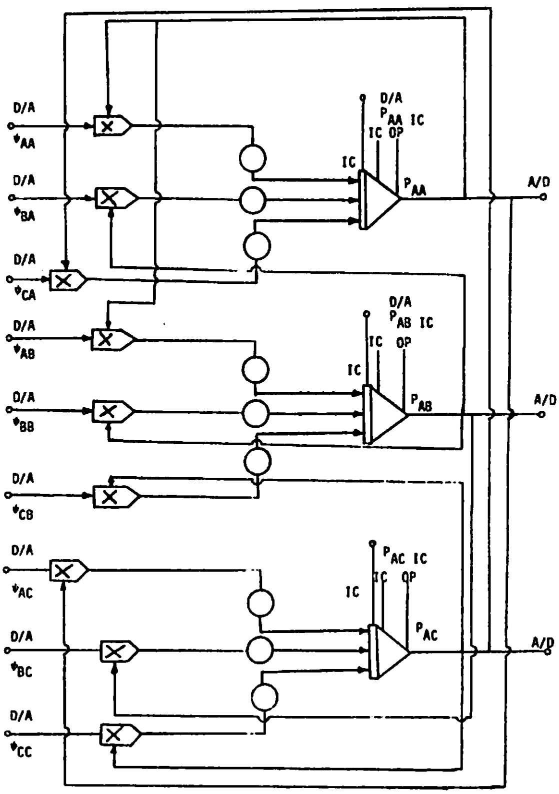


Figure 4. Sample analog diagram for the Kolmogorov System.

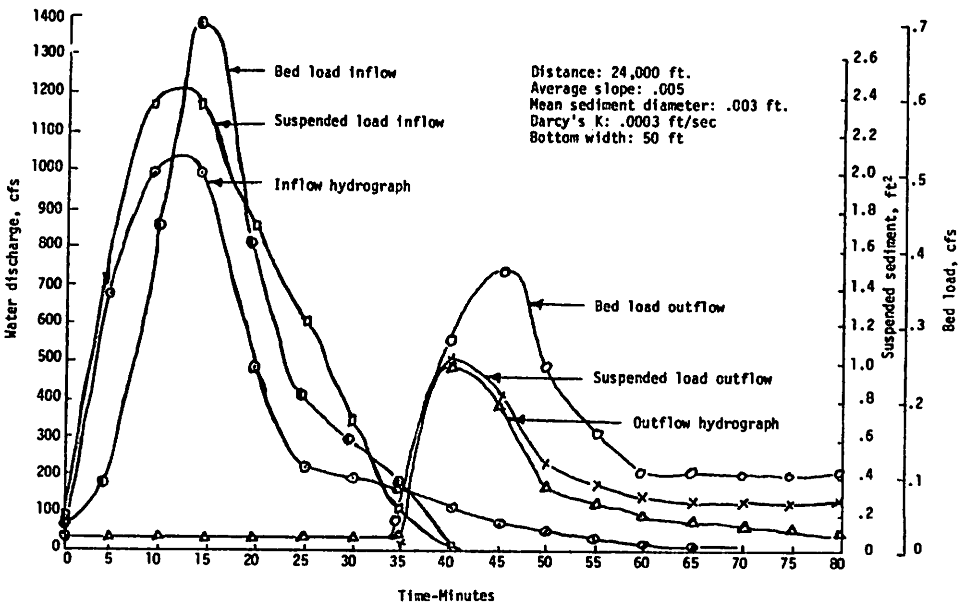


Figure 5. Inflows and simulated outflows

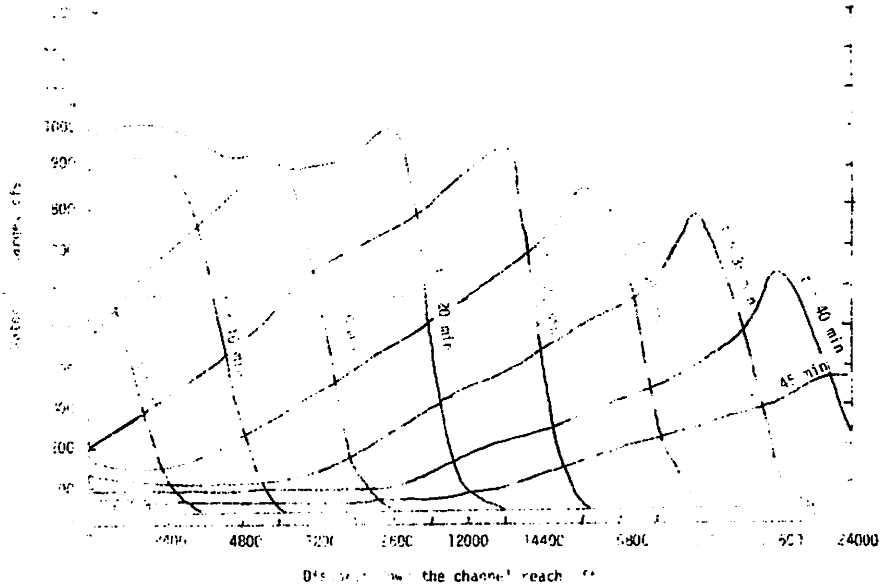
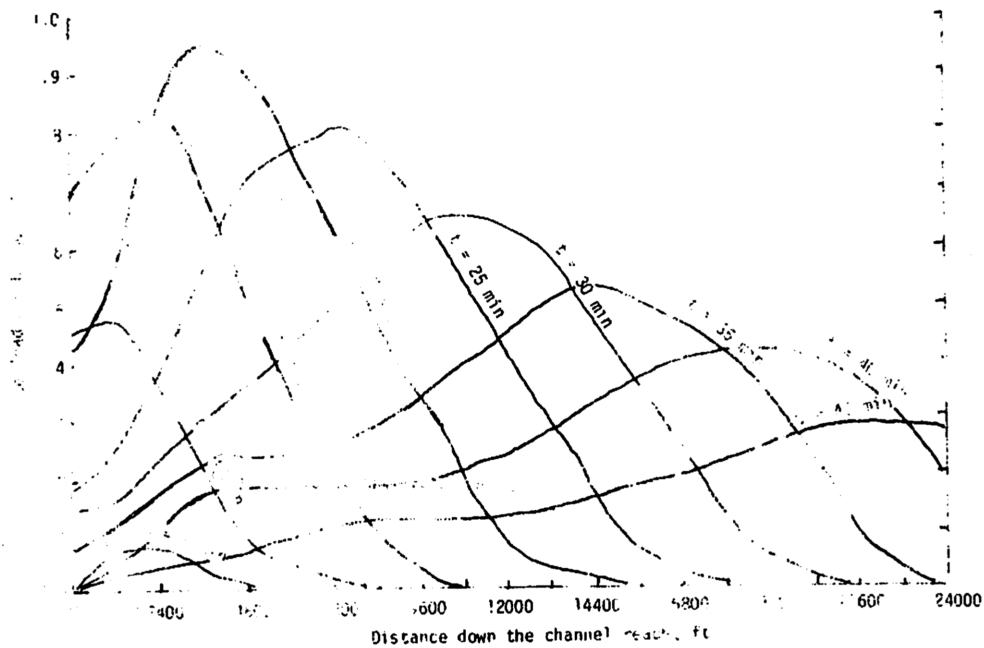


Figure ... ater discharge profil...



... Bed load discharge profiles

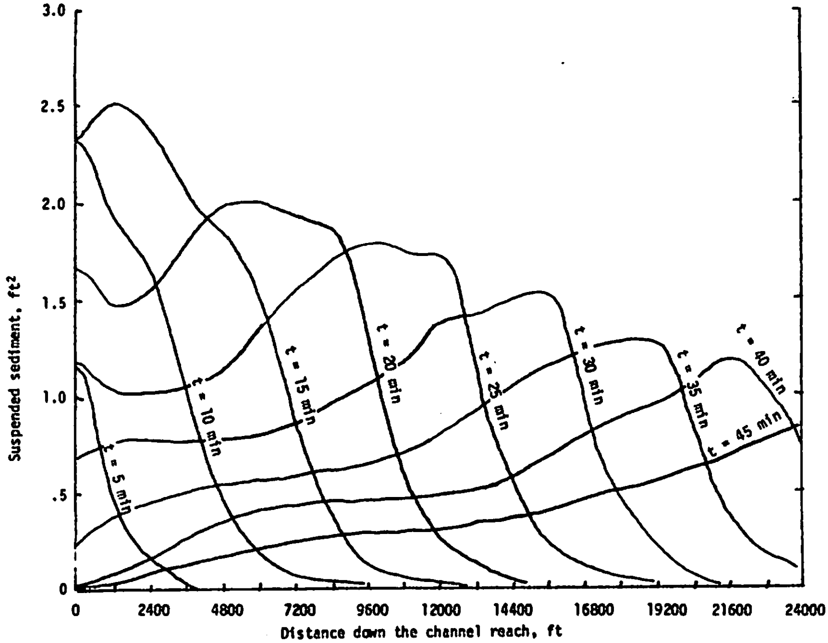


Figure 8. Suspended load profiles

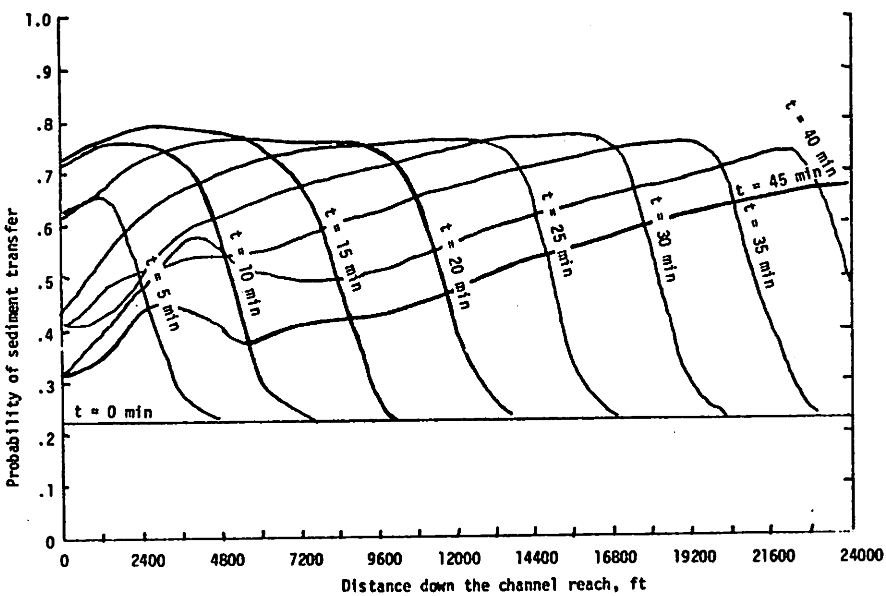


Figure 9. Profiles of the probability of the transfer of sediment from the immobile bed state to bed load state, P_{CB}.