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Design and operation of engineering systems using regularized stochastic decomposition

Walid E. Elshorbagy & Kevin E. Lansey
Department of Civil Engineering and Engineering Mechanics, University of Arizona, Tucson, Ariz., USA
Diana S. Yakowitz
Southwest Watershed Research Center, Agricultural Research Service, USDA, Tucson, Ariz., USA

ABSTRACT: Many engineering systems are affected by uncertainty in future demands or inputs. Decisions regarding their design, however, typically must be made in the present. Two-stage stochastic programming can consider this type of problem but, in the past, procedures to fully incorporate the uncertainty have come with a high computational cost. New algorithmic developments, such as Regularized Stochastic Decomposition (RSD), now allow more complex systems to be considered. This paper provides an overview of the RSD method and its extensions and demonstrates the application of two-stage stochastic programming to two water resources problems with different problem structures and types of uncertainty.

1 INTRODUCTION

Design and analysis of engineering systems usually involve many uncertainties. Oftentimes, these uncertainties are neglected because they are not known, their values are low, or the uncertain parameters play an insignificant role in the process. However, when uncertain parameters can significantly affect the design, their uncertainties must be taken into account.

One way to account for these uncertainties in mathematical programming models during design is to use a probabilistic representation instead of the best estimates of the uncertain coefficients. Some versions of this type of model, were introduced in the late 1950's by Charnes and Cooper (1959). One such approach is chance constrained programming that is widely used for design problems.

Many engineering applications, however, involve design and operation decisions, such as irrigation canal layout and water allocation. These problems can be formulated as two-stage programs that consider the future system operations when making design decisions. The uncertainties are accounted for by assigning a probability distribution to the uncertain coefficients.

Although this approach is well known in the area of operations research, it has not been broadly applied to hydraulic and water resources engineering problems. This paper describes and applies one of the more recently developed algorithms to solve two-stage stochastic linear programs with recourse. Higle and Sen (1991) introduced the Stochastic Decomposition (SD) approach that combines the strengths of decomposition based algorithms and stochastic gradient methods.

SD produces a piecewise linear approximation of the objective function, then solves one subproblem and one master program at each algorithm iteration. A major problem with SD is that the master program progressively increases in size of as a result of a new cut (constraint) being added during each iteration. This problem can result in severe computational effort, especially for large problems with many random parameters.

Recently, Yakowitz (1994) introduced a quadratic regularizing term in the master program that limits the movement of the Master problem solutions so that the function estimates remain adequate. In this procedure, the size of the master program can be limited by introducing a cut-dropping scheme similar to that given in Mifflin (1977) and Kiweil (1985).

Regularized Stochastic Decomposition (RSD), has been applied to solve simple engineering applications under limiting conditions. The present work employs this approach and modifies it so that it can be applied to a wider and more practical range of applications. Two applications are presented to demonstrate its capabilites.

2 BACKGROUND

Two-stage stochastic LP with recourse problems have a first-stage set of decisions that must be made at present. The set of second-stage variables are determined in the future based on the actual future conditions while satisfying restrictions resulting from the first-stage decisions.

A general formulation of this type of problem is:

$$Min\ f(x) = cx + E_{\bar{\omega}}[Q(x,\bar{\omega})] \quad (1)$$

$$s.t. \quad x \in X \subset R^{n1} \quad (2)$$

$$where \quad Q(x,\bar{\omega}) = Min\ qy \quad (3)$$

$$s.t. \quad Wy = \bar{\omega} - Tx \quad (4)$$

The problem consists of: [1] a first-stage objective function, cx, with its associated n1 first-stage decision variables, x, and m1 first-stage constraints, and [2] a second stage objective function $Q(x,\varpi)$, with second-stage solution y of n2 variables, and m2 second-stage constraints based on some observation ϖ. The random vector, ϖ, is defined on a probability space (Ω,A,P) where Ω is a compact set. The distribution probability function, F_ϖ, is associated with ϖ, and $E_\varpi[.]$ is the mathematical expectation with respect to ϖ. The set of feasible first-stage decisions, X, is assumed to be convex and bounded. With these conditions, the total objective function is a piecewise linear convex function of x.

3 SOLUTION ALGORITHM

At each iteration of the RSD algorithm, one Master Program and one second stage Subprogram is solved. The Master program consists of the first stage objective and a piecewise linear approximation of the expected second stage objective. First stage constraints include cutting planes developed in the second stage problem.

The second stage program is solved at the present best Master program solution for the optimum future decisions at a realization of the uncertain coefficients. The new candidate solution is compared to the current solution and accepted or rejected. Iterations continue until the algorithm terminates by satisfying appropriate stopping rules. The steps required in the algorithm are briefly listed below:

Step 0. Initialize with a feasible current first stage decision (e.g., optmal solution using the expected values of random variables) and set the candidate first stage decision equal to the current first stage decision.

Step 1. Randomly generate a single observation of all random variables according to their distributions.

Step 2. Solve the second stage problem that results from Step 1 at the candidate first stage decision and save the solution.

Step 3. Estimate the cutting plane at the current first stage decision to be added to the Master program using the current and past solutions to the second stage problem.

Step 4. Determine if the objective estimate at the candidate solution is significantly lower than the estimate of the objective function at the current first stage decision. If so, the candidate becomes the current solution.

Step 5. Update, re-evaluate, or eliminate past cutting planes and solve the current Master program.

Step 6. Determine if the stopping criteria are met. If so, stop. Otherwise, return to Step 1.

The algorithm was coded in Fortran so that any two-stage stochastic problem can be solved without changing the code. GRG2 (Lasdon. 1985), a nonlinear programming model, was used to solve the master problem at each iteration. GRG2 applies the generalized reduced gradient method as a basis for solving the NLP.

4 RSD EXTENSIONS

The RSD approach, to date, has been limited to solving linear two-stage problems with uncertainty only in the RHS of the second-stage constraints. Since many engineering problems behave in a nonlinear manner, the algorithm was modified to handle nonlinearity of the first-stage objective function. In this case, the convexity assumption is violated. Therefore, global optimality of the optimal solution is no longer guaranteed.

However, local optimal solutions are often adequate in engineering practice. Prudent selection of the initial point can potentially improve the solution and bring it closer to the global optimal solution.

The nonlinearity in the first-stage objective is introduced by including the nonlinear function, $g(x_k)$, in the objective of the master program. Cuts are identified in the same manner as the linear first-stage problem, however, they now only approximate the second-stage stochastic function.

The second limitation to practical application of SD and RSD is the requirement of deterministic coefficients in the second-stage objective function. These terms can include future revenues and/or prices that may also be uncertain. The algorithm was successfully modified to handle these uncertainties. However, computational problems during implementation point out the difficulty of its practical application. Details of the RSD and its extensions can be found in Elshorbagy et al (1995). Two applications are presented in the next sections to demonstrate the utility of RSD to water resources and hydraulics problems.

5 REGIONAL WATER SUPPLY PLANNING

Consider a region which has two communities. Each community has demands for both potable water for municipal use, and reused water for irrigation and other purposes. The goal is to size the water supply facilities required to satisfy consumer demands over a 20-yr period. Potable water demands can be met by direct supply from the aquifer and/or treated water from the water treatment plant which is supplied from a surface source (Figure 1). The demands of reused water can be also met from direct supply from the aquifer or from a tertiary treatment plant which is supplied from a secondary wastewater treatment plant. The aquifer is recharged through a infiltration basin system with water from the river or the wastewater treatment plant after secondary treatment.

The planning problem is to determine the design capacities of the recharge basin, water treatment plant, secondary wastewater treatment plant, and tertiary treatment facility. These decisions represent the first-stage decision variables in the two-stage formulation. The second-stage variables represent the water allocations (in million gallons per day, mgd) from

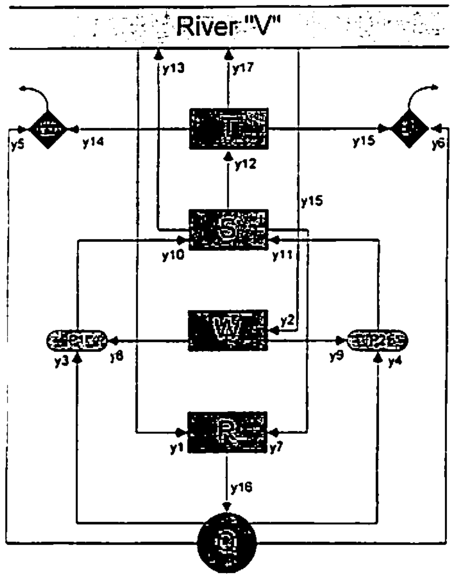

Figure 1: Regional water supply system

the supply facilities to different users during different time periods. The variables y_1 to y_{17}, shown on the system outline of Figure 1, are the second-stage operation variables for the first period. The total number of the second-stage variables for the two periods is 34.

The first-stage objective function represents the present construction cost of the four supply facilities. The second-stage objective represents the expected value of the uncertain future operation costs. These costs result from treating and pumping water during the two 10 year periods. They are assumed to be linear functions of the amounts of treated and delivered water, respectively, and were brought to a present value.

The structure of the water supply planning problem, given in the two-stage formulation, can be written as:

$$\underset{(x,y)}{MIN} \quad \sum_{l=1}^{4} c_l * x_l + 6.145 * 365 * \\ E[\sum_{r=1}^{17} q_r^1 * y_r^1 + 0.386 * \sum_{r=18}^{r=34} q_r^2 * y_r^2] \\ + q_e * [\sum_{\xi=1}^{2} [eq^{\xi} + \sum_{v=1}^{2} (ep_v^{\xi} + eu_v^{\xi})] \tag{5}$$

<u>Subject to</u>

First-stage constraints

$$x_l \geq 0 \qquad l \in \{1,4\} \tag{6}$$

Second-stage constraints (For $\xi = 1,2$)

[1] Canal Capacity

$$\sum y^{\xi}_{\to x_l} \le x_l \qquad l \in \{1,4\} \tag{7}$$

[2] Water Availability

$$\sum y^{\xi}_{-V} \le AV^{\xi} \tag{8}$$

[3] Potable and Reuse Demands

$$\sum y^{k}_{\to P} + ep^{\xi}_{v} \ge DP^{\xi}_{v} \qquad v=1,2 \tag{9}$$

$$\sum y^{\xi}_{\to U} + eu^{\xi}_{v} \ge DU^{\xi}_{v} \qquad v=1,2 \tag{10}$$

[4] Aquifer Storage

$$QI^{\xi} + \sum y^{\xi}_{-Q} - \sum y^{\xi}_{-Q} + eq^{\xi} \ge QS^{\xi} \tag{11}$$

[5] Quality of Reuse Demands

$$y^{\xi}_{Q\to U_v} \ge PCU * DU^{\xi}_{v} \qquad v=1,2 \tag{12}$$

[6] Quality of Potable Demands

$$y^{\xi}_{W\to P_v} \ge PCP * DP^{\xi}_{v} \qquad v=1,2 \tag{13}$$

[7] Temporal Continuity

$$QI^{\xi} + \sum y^{\xi}_{-V} + \sum_{v} (ep^{\xi}_{v} + eu^{\xi}_{v}) + eq^{\xi} \ge (1+loss_{avg}) * \sum_{v} DP^{\xi}_{v} + DU^{\xi}_{v} \tag{14}$$

[8] Mass Continuity

$$(1-loss_j) * \sum_{j} y^{\xi}_{\to j} = \sum_{j} y^{\xi}_{\to j} \qquad j \in \{R,W,S,T,P1,P2\} \tag{15}$$

where x_i is the design capacity of the supply units with x_1, x_2, x_3, and x_4 being capacities of the recharge basin, water, secondary, and tertiary treatment plants, respectively. q_r^1 is an objective function coefficient related to the allocation, y_r^1 (the superscript 1 means the first period), and depends on its treatment and pumping costs. q_e is a unit price of the penalty water used to maintain feasibilty. The first-stage constraints are only simple bounds to maintain non-negative values of the capacities. The subscript of y on the second-stage constraints identifies the allocated water. For example, $y_{\to U}$, defines all y's entering unit U, and $y_{Q\to U1}$ defines the allocated water from unit Q to unit U1. The second-stage constraints, are divided into eight groups as follows:

[1] Capacity constraints ensure that the total delivered amount of water to any unit during any time period, ξ, will be less than the capacity of the unit.

[2] River Availability constraints ensure that the available water in the river, AV, exceeds the amount diverted to the system during any time period.

[3] Demand constraints guarantee that the potable demands, DP, and the reuse demands, DU, are satisfied for the two communities during any period, ξ. ep, and eu, are external penalty water required to maintain feasibility during random generated constraints which may cause the demand to exceed the supply.

[4] Aquifer Storage Constraints assure that the amount of water stored in the aquifer at the end of each period is greater than a pre-specified reserve amount, QS. The amount of stored water equals the initial storage, QI, plus entering water minus withdrawn water plus external penalty water.

[5] Reuse Quality Constraints maintain a pre-specified ratio of the total reuse demands, PCR, to be direct supply from the aquifer.

[6] Potable Quality Constraints maintain a pre-specified ratio of the total potable demands, PCP, to be delivered from the water treatment plant.

[7] Temporal Continuity Constraints insure that all demands and losses are met using true sources of water.

[8] Mass Balances Constraints preserve the mass balances at different nodes and accounting of their losses. The nodes of concern are the supplying units and the two nodes of potable demands (P).

The total number of second-stage constraints in this problem is forty two. The stochastic parameters in the right hand side (RHS) of the second-stage constraints are AV, DP, and DU. The number of independent random parameters considered in this case is 10. Stochasticity in the treatment costs of the four supplying units along with the pumping costs, are also considered which represent eight independent random parameters.

5.1 *Results and discussion*

5.1.1 *Linear first-stage objective function and stochastic RHS*

The design capacities for this condition were obtained in 15.75 hours using a SPARC-station LX computer system. The four capacities obtained using this design were larger than those of a deterministic design but provided an overall 5% improvement in the total objective function.

5.1.2 *Non-linear first-stage objective function and stochastic RHS*

To introduce nonlinearity, a power function was used as the first-stage objective function. Designs were determined for two values of the power function exponent. Facility components were enlarged when a concave objective was used (power coefficient = 0.80) while they were reduced in the convex case (power coefficient = 1.5). This result is expected due to the economy of scale with a concave objective. In both cases, the optimal solution was improved compared to a deterministic design with the same objective with 11 and 23% gain for the concave and convex objectives, respectively.

5.1.3 *Linear first-stage objective function and stochastic second-stage objective function*

The solution for this case was identical as that obtained from a deterministic model. Thus, the variability in the objective coefficients had no effect on the first-stage decisions for this example which is not expected in all cases.

This solution was found after a large number of algorithm iterations that required significant computation time. The cause of this problem was that the number of vertices was growing with the number of iterations. The vertices are related to the number of independent dual variables that are possible in the subproblem. Stochastic RHS problems have a finite number of these vertices, so convergence is possible. The growth in dual variables, however, was not expected nor reported in the literature when stochastic objective coefficients are considered. Numerous unsuccessful alternative schemes were developed to alleviate this problem.

6 IRRIGATION CANAL SYSTEM DESIGN

Canal capacities of an irrigation system have a great impact on future farm revenues under different operational conditions. Allocation models, developed to plan for the most economical way of distributing water to crops at different growth stages, are all constrained by predefined canal capacities. Using RSD, it is possible to determine the best canal capacities during the initial design stage while considering varying operation conditions.

A demonstration system consisting of 9 canals, 6 fields, and one source of water is considered. The planning horizon was 10 years with two 6-month growing seasons in each year. Figure 2 shows the outline of a symmetrical branch system of canals and the crops grown in each field during each season.

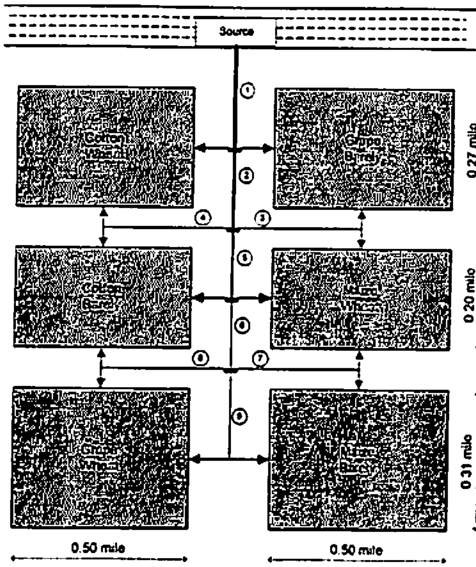

Figure 2: Application irrigation system

The first stage represents the canal construction cost. Linear construction cost functions were applied and reflect their dependence on the canal length and the excavation cost. The second stage is the negative of the net benefit of different crop yields from future time periods. Two sets of constraints corresponding to the two stages are also included. The first-stage constraints are simple bounds to maintain positive canal capacities within a defined range. The second-stage constraints are divided into three types; [1] crop-demand requirements; [2] canal-capacity limitations; and [3] water-availability constraints. This formulation can be stated as:

$$Min \sum_{i=1}^{nc} \alpha_i c_i - E \; \{ \sum_{k=1}^{nk} \sum_{j=1}^{nf} [\overline{Py_{jk}} * A_j * Y_{jk} - \overline{Px_k} * \sum_{l=1}^{nx} (NFX_{jl} * A_j * x_{lk})] \} \quad (16)$$

subject to:
First-stage Constraints

$$C_l \leq C_i \leq C_u \qquad \forall i \quad (17)$$

Second-stage Constraints

$$\sum_{l=1}^{nx} NXF_{jl} * x_{lk} \leq \overline{D_{jk}} \qquad \forall j,k \quad (18)$$

$$\sum_{l=1}^{nx} [(1.+conv.\ loss\ ratio)^{(LR_{li}-1)} * x_{lk}] < c_i \qquad \forall i,k \quad (19)$$

$$\sum_{l=1}^{nx} [(1.+ conv.\ loss\ ratio)^{(LR_{li}-1)} * x_{lk}] < \overline{AW_k} \qquad \forall k \quad (20)$$

where C_i is the i-th canal capacity of nc canals; α_i is a construction cost-capacity coefficient; A_j is the area of field j of nf fields; Py_{jk} is the commodity price per harvested crop unit weight ($/unit weight); Y_{jk} is the crop yield in field j harvested during period k (unit weight/unit area); Px_k is the cost of water during period k ($/unit volume); X_{lk} is the lth allocated water of nx allocations (unit depth); and NFX_{jl} is an identifier flag. NFX equals 1 or 0 defining that allocation l is or is not connected to field j, respectively. C_l and C_u are lower and upper bounds for the canal capacities. D_{jk} is the crop demands of field j during period k and AW_k is the amount of available water during period k. Since D_{jk}, AW_k, Py, and Px are future parameters, they are treated as random variables and denoted with overbars. LR_{li} is the order of the allocation l with respect to canal i. For example, $LR_{li} = 1$ means that the lth allocation has a 1st order rank to the ith canal, or the lth allocation is directly connected to canal i where no conveyance loss is considered. If the allocation is upstream of the canal so that it does not contribute anything to the flow in the canal, then the whole term corresponding to that allocation is omitted.

The crop yield (Y_{jk}) is determined using the FAO crop response function (Doorenbos and Kassam, (1979)).

$$Y_{jk} = Ym_{jk} [1 - Ky_{jk}[1 - \frac{IE_{jk}}{ETm_{jk}} * \sum_{l=1}^{nx} NFX_{jl} * x_{lk}]] \quad (21)$$

where Ym_{jk} is the maximum yield of field j harvested during period k (Mg/ha); Ky_{jk} is the FAO yield response factor in fraction for field j; ETm_{jk} is the maximum evapotranspiration of field j during period k (mm); and IE_{jk} is the irrigation efficiency (fraction) for field j during period k.

6.1 *Results*

The stochastic optimization model has two types of uncertain parameters which appear in the right hand side of the model constraints; crop demands and available water. The twelve objective function coefficients (crops selling prices), were also considered as uncertain. It was assumed that they were described by normal distributions with different coefficients of variation (CV). The effect of each on canal capacities and future revenues was evaluated using the RSD approach for different levels of parameter uncertainty. All runs were made on a SPARC station LX. The results are compared with an optimal deterministic solution which was developed using the mean parameter values.

6.1.1 *Available Water*

Sufficient amounts of available water during the two periods (condition of zero deficit level) were assumed to follow normal distribution with mean values equal to 3.22 and 2.91 millions of cubic meters/season, respectively. These flows were determined the water demand in a deterministic optimization problem with unlimited available water.

Stochastic problems were solved with different coefficients of variation (CV) for the random available water with the mean values noted above. In all cases, the canal capacities were identical to those obtained in the deterministic design.

A more important problem is balancing the canal capacity under shortage conditions. Thus, a like analysis was repeated for different water deficit levels. In these cases, the mean available water was computed by decreasing the sufficient available water by the deficit level percentage.

Table 1 lists the percentage increase of the stochastic solution in the net revenues compared to the deterministic result for different deficit levels and available water coefficient of variations. Canal capacities were the same for deficit levels of 10% and 25% and were similar to the design obtained from deterministic design with zero deficit level. For the 50% deficit level, however, the design capacities using the RSD approach dramatically changed, as did the percentage increase in revenues. The last observation points out the importance of using the stochastic approach in design when a significant shortage of available water can be expected.

Table 1. Percent increase in net revenues between deterministic and stochastic irrigation system designs

	Coefficient of Variation, CV			
Deficit level	0.25	0.5	0.75	1.0
10%	2.4	2.9	3.3	3.6
25%	4.9	8.0	9.9	11.2
50%	4.7	10.9	17.3	22.4

6.1.2 *Crop Demands*

The influence of the variability of crop water demand on the canal design capacities was also evaluated. This demand is related to the potential evapotranspiration and the irrigation efficiency. Continuous normal distributions for crop demands were assumed and the deterministic demands were assumed to be the mean values. Canal capacities from the deterministic and stochastic designs (CV=0.50) and showed a slight increase in revenues using capacities obtained from the stochastic design.

Another design was carried out when both the available water (of 25% deficit level) and the demands, were considered stochastic. The percentage increase in net revenues compared to the deterministic design was 9.6%. The percentage increase in net revenues when only stochastic water availability was considered was 8.5%. The minor change between the two cases indicates that the canal capacities are not very sensitive to the variabilities of crop demands for this system.

6.1.3 *Influence of the construction cost coefficient*

Another set of cost coefficients was chosen to evaluate the influence of these coefficients on the canal capacities, as well as the revenues obtained using different design approaches. The inital cost coefficients were multiplied by a factor of 5 and new stochastic and deterministic designs were computed at the 50% deficit level for different CV's of available water.

The improvement over the deterministic solution was smaller compared to results using the lower cost coefficients shown in Table 1. The percentage improvements were 2.13, 5.36, 9.19, and 13.03 for CVs of 0.25, 0.5, 0.75 and 1.0, respectively. Since the canal sizes are decreased with larger cost, the high available flows and their benefits are not available so the stochastic and deterministic returns become closer to each other.

6.1.4 *Stochastic Objective Coefficients*

Since RSD had poor convergence in the water supply problem, an alternative method, known as the L-shaped algorithm (Van Slyke and Wets, 1969) was used to solve this problem. However, a large sample of linear prgograms must be solved during each iteration of the L-Shaped algorithm, so that all possible outcomes are covered in the stochastic second stage. Three values were defined for each random objective function coefficient. Each coefficient was randomly generated from a continuous distribution then approximated to the nearest value of the three discretized values. For the discretized distributions, the number of possible outcomes is tremendous (3^{12}=531441 outcomes). As an approximation, a finite sample with reasonable size is sufficient in most situations. For the current irrigation system analysis, by examining different sample sizes, it was determined that this condition was reached after 4000 outcomes.

At the zero deficit level and all coefficients of variation, the design capacities of the canals were identical to the deterministic design with the mean coefficient values.

The available water was then lowered to 25%

and 50% deficit levels and the problem was solved for different coefficients of variation (CV=0.25, 0.50, 0.75, and 1.0). The stochastic designs were compared to the determinstic solution as a percentage increase in net revenues (Table 2). In the deterministic design case, the expected return was computed by Monte-Carlo analysis with the same CV's of the objective function coefficients and a sample size of 4000. Although not listed, the design capacities changed considerably from one CV to another. The improvements are significant which demonstrates the sensitivity of the design to net crop selling prices.

Table 2. Percent increase in net revenues for stochastic objective coefficients and different CVs

	Coefficient of Variation, CV			
Deficit level	0.25	0.5	0.75	1
25%	0.1	1.1	3.3	6.3
50%	0.6	4.6	12.5	23.3

6.1.5 *Stochastic objective and RHS coefficients*

The final analysis treated the available water and the objective function coefficients as random. As in the case of only stochastic objective coefficients, the parameter variability at the zero deficit level had no effect on the recommended canal capacities. This conclusion was consistent for all CVs.

At 25% deficit levels, design capacities differed from the deterministic design. However, for CV >= 0.50, the capacities did not change and the system design was identical to zero deficit level case. The gain in net revenues increased with CV until it reached a maximum value of 9.8% at CV=0.50 after that CV the gain decreased. The decrease at higher CV's is attributed to the increase of high level of variability. Similar results and behavior were observed for the 50% deficit level.

Stochasticity in both the objective and RHS may result in a non-convex problem and non-global optimal solutions. Therefore, the L-shaped model was executed using different starting points. All initial points resulted in the same solution which indicates that the solutions are likely globally optimal.

7 CONCLUSIONS

Engineered systems that face an uncertain future are design/operation problems. Two-stage stochastic optimization has been demonstrated to be a useful tool in examining these types of problems. A promising solution algorithm is regularized stochastic decomposition. In this paper, RSD has been shown to be capable of solving general water resources problems. The benefits of the approach are clearly seen in the improved returns from the irrigation canal design and water supply planning studies.

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