Unsteady Free-surface Flow Problems

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1. Introduction

Unsteady, free-surface flow as overland or open channel flow is the most dynamic part of the response of a watershed to precipitation. Floods are almost invariably caused by direct surface runoff from rainfall, melting snow or a combination of the two. Therefore, designs for flood control structures or non-structural measures to reduce flood damage must utilize estimates of flood peak rates and frequencies. On the many small agricultural and urban watersheds that have no discharge records the engineer must utilize mathematical models along with rainfall data to make such estimates. In this paper we will consider one important part of hydrologic models — overland flow.

Although the term 'overland flow' is frequently used in reference to very shallow flow over plane surfaces, a less restrictive definition will be used herein. Overland flow will refer to all those flows for which the model discussed in this paper is a useful abstraction. It will include thin sheet flow over plane surfaces and may also include flows over rilled and irregular surfaces or flow in small channels. No precise definition of the boundary between overland flow and channel flow will be given because such a separation is purely operational.

2. The equations of spatially-varied, unsteady flow over a plane

The equations of spatially-varied, unsteady flow over a plane describe many of the important aspects of overland flow. The problem under consideration is shown in Figure 1. A plane of unit width, length $L_0$ and slope $S_0$ is receiving rainfall at a rate $i(x,t)$ per unit area. Water is infiltrating at a rate $i_r(x,t)$. The net rate of lateral inflow is

$$q(x, t) = i(x, t) - i_r(x, t)$$ (1)

Flow is assumed to be one-dimensional and the dependent variables are the local

Rainfall \( i(x,t) \)

\[ \frac{\partial b}{\partial t} + \frac{\partial (Ub)}{\partial x} = q(x, t) \]

(2)

and the momentum equation is

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial b}{\partial x} = g(S_o - S_f) - \frac{U}{b} q(x, t) \]

(3)

where \( S_f \) is the friction slope. This formulation assumes that over pressure introduced by rainfall is negligible and that the velocity component of the rainfall in the \( x \) direction is zero. The obvious assumptions that the sine of the slope angle, \( \theta \), is approximately equal to the tangent and that the velocity distribution coefficient \( \beta \) is equal to one, are also included.

To investigate the important properties of this model we will consider a very simple case. We will assume that the plane is impervious and that the rainfall rate is not a function of \( x \) or \( t \).

By writing equations 2 and 3 in dimensionless form, the number of parameters can be reduced from five to two with obvious advantages in the graphical portrayal of results.

Experience has shown that the following normalizing quantities are useful.

Normalizing lengths:

\[ L_0 = \text{length of plane} \]

\[ H_0 = \text{normal depth at } x = L_0 \text{ and a flow rate of } qL_0. \]
Normalizing velocity:

\[ U_0 = \frac{qL_0}{H_0} = C \sqrt{H_0 S_0} \]  \hspace{1cm} \text{(if the Chézy relationship is used).}

Define the following dimensionless variables

\[ \frac{b^*}{H_0} ; \quad \frac{U^*}{U_0} ; \quad \frac{x^*}{L_0} ; \quad \frac{t^*}{L_0} \]

and \( q^* = \frac{q}{q_0} = 1 \), where \( q_0 \) is the maximum lateral inflow rate.

By substituting the dimensionless variables into equations 2 and 3 we obtain:

\[ \frac{\partial b^*}{\partial t^*} + \frac{\partial (U^* b^*)}{\partial x^*} = 1 \]

\[ \frac{\partial U^*}{\partial t^*} + U^* \frac{\partial U^*}{\partial x^*} + \frac{1}{F_0} \frac{\partial b^*}{\partial x^*} = \frac{S_0 L_0}{H_0 F_0^2} \left( 1 - \frac{U^{*2}}{b^*} \right) - \frac{U^*}{b^*} \]

where

\[ F_0 = \frac{U_0}{\sqrt{gH_0}} \quad \text{(Froude number)} \]

The dimensionless parameter \( S_0 L_0 / H_0 F_0^2 \) can be interpreted as half the ratio of the total potential energy drop along the plane to the normal energy head at the downstream boundary. Suitable boundary conditions for this problem are the dry initial condition:

\[ b^*(x^*, 0) = U^*(x^*, 0) = 0 \]

The condition of zero flux at the upper boundary

\[ U^*(0, t^*) = 0 \]

and the critical depth lower boundary condition:

\[ U^*(1, t^*) = C^*(1, t^*) \]

if

\[ U^*(1 - \Delta x^*, t^*) \leq C^*(1 - \Delta x^*, t^*) \]

where \( C^* = \sqrt{(b^*)/F_0} \) and \( \Delta x^* \to 0 \). No lower boundary condition is required if the flow is supercritical.

3. Solutions of overland flow equations

Numerical solutions of equations 4 and 5 reveal interesting properties of the rising hydrograph. As the parameter \( S_0 L_0 / H_0 F_0^2 \), which we shall now denote by \( k \), the kinematic flow number, increases, the solution converges very rapidly toward that for \( k = \infty \). The effect of varying the parameter \( k \) while holding \( F_0 \) constant is shown in Figure 2(a). For \( F_0 = 1 \) the maximum error between the rising hydrographs for \( k = 10 \) and \( k = \infty \) is of the order of 10%. The effect of varying \( F_0 \) while \( |k| \) is held constant is shown for three values of \( k \) in Figure 2(b), (c) and (d). For \( k = 1 \), the
rising hydrographs are very complicated and obviously cannot be represented by a single dimensionless hydrograph. However, as the parameter $k$ increases, the rising hydrographs for different values of $F_0$ approach the dimensionless hydrograph for $k = \infty$.

The effect of varying $k$ on the steady state celerity and velocity profile is shown in Figure 3. As $k$ increases, both profiles converge on the profile for $k = \infty$. 
What is the physical significance of the $k = \infty$ case? An examination of equation 5 shows that if all terms in the equation are divided by $k$, the momentum equation reduces to the following as $k$ approaches infinity

$$b^* = U^*^2$$

This is the dimensionless Chézy equation. Therefore, there is a unique relationship
between depth and discharge, and the depth is the normal depth for uniform flow at that discharge. When \( k \) is large the solution to equations 4 and 5 can be closely approximated by the solutions to equations 4 and 7. This is the kinematic wave approximation which has been described in detail by several investigators (Lighthill and Whitham, 1955; Iwagaki, 1955).

A few quick calculations will show that the kinematic approximation is excellent for most cases of overland flow. For example, the parameter \( k \) is 7.4 for a plane 20 feet long, with a slope of 0.00025, Manning's \( n \) of 0.01 and a lateral inflow rate of 5 inches per hour. A plane this short, flat and smooth would be highly unusual in hydrologic applications.

4. Kinematic wave solutions for simple watershed geometry

The kinematic wave equations have an important advantage over the shallow water equations for describing overland flow; analytic solutions are possible for simple geometries. A great deal of insight into the phenomenon of overland flow can be gained by studying these solutions.

Figure 3. Celerity and velocity profiles: Variation with \( k \) (From Woolhiser and Liggett, 1967)
For overland flow on a plane as shown in Figure 1, the dimensional kinematic wave equations are
\[
\frac{\partial b}{\partial t} + \frac{\partial (Ub)}{\partial x} = q(x, t)
\]
and
\[
S_0 = S_f
\]
Equation 9 can be written in the convenient Darcy—Weisbach form
\[
U = \sqrt{\left(\frac{8g}{f} \, S_0 b\right)}
\]
where \( f \) is a friction factor. In solving for hydrographs from simple watershed geometries we will assume the following parametric relationship
\[
U = \alpha b^{m-1}
\]
We shall examine appropriate friction relationships and their implications in a later section.

Equation 11 can be substituted into equation 8 to obtain a partial differential equation with one dependent variable
\[
\frac{\partial b}{\partial t} + \alpha \frac{\partial b^m}{\partial x} = q(x, t)
\]
Equation 12 which is a partial differential equation can be transformed into the characteristic form in the following manner. Along the solution surface the following expression for the total derivative holds:
\[
\frac{\partial b}{\partial t} = \frac{\partial b}{\partial x} \frac{dx}{dt} + \frac{\partial b}{\partial x} \frac{dt}{dx}
\]
Both equations 12 and 13 must hold simultaneously so in matrix notation we have
\[
\begin{bmatrix}
1 & \alpha mb^{m-1} \\
dt & dx
\end{bmatrix}
\begin{bmatrix}
\frac{\partial b}{\partial t} \\
\frac{\partial b}{\partial x}
\end{bmatrix}
= \begin{bmatrix}
q(x, t) \\
\frac{db}{dt}
\end{bmatrix}
\]
By equating the square matrix on the left side to zero we obtain the equation of the characteristic ground curve
\[
\frac{dx}{dt} = \alpha mb^{m-1} = mU
\]
The comparability condition (which ensures an infinite number of solutions of the simultaneous equations) yields
\[
\frac{db}{dt} = q(x, t)
\]
which holds to the direction specified by equation 15.
If we consider the simple case of spatially invariant rainfall on an impervious surface beginning at a constant rate, \( i \), at time \( t = 0 \) and continuing until \( t = \infty \), we can solve the above two equations to obtain the solution to the rising hydrograph. Consider the characteristic emanating from \( (x_0, t_0) \) and intersecting the line \( x = L_0 \) at time \( t' \), as shown in Figure 4. If the plane is initially dry, i.e. \( b = 0 \) at \( t = 0 \), integration of equation 16 gives

\[
b = it
\]

or

\[
Q = Ub = \alpha (it)^m
\]

The presence of the upstream boundary along which \( b = 0 \) is not detected at \( x = L_0 \) until the characteristic originating at \( (0,0) \) arrives. This time can be found by solving equation 15. Substituting the expression for \( b \) from equation 17 into equation 15

\[
\frac{dx}{dt} = \alpha m (it)^{m-1}
\]

\[
x = \alpha^{m-1} t^m + C
\]

The constant of integration is zero, because when \( x = 0, t = 0 \). Therefore

\[
t_e = \left( \frac{L_0}{\alpha i^{m-1}} \right)^{1/m}
\]

The equation for the response of a plane to a step input is therefore

\[
Q = \alpha (it)^m; \quad 0 \leq t \leq \left( \frac{L_0}{\alpha i^{m-1}} \right)^{1/m}
\]

\[
Q = iL_0; \quad \left( \frac{L_0}{\alpha i^{m-1}} \right)^{1/m} < t < \infty
\]
the line \( t = 0 \) and the line \( x = L_0 \) is uniform and unsteady. Outside this area the flow is steady but non-uniform.

The hydrograph of the recession from equilibrium can be found by assuming that at some time, \( t_0 > t_e \) the inflow ceases. Solution of equation 16 gives

\[
b = b_0
\]

which is valid along the characteristic

\[
x = x_0 + \alpha mb_0^{m-1}(t - t_0)
\]

where \( b_0 \) is the equilibrium depth at \((x_0, t_0)\). The expression for the equilibrium depth

\[
b_0 = \left(\frac{i x_0}{\alpha}\right)^{1/m}
\]

can be substituted into equation 24 giving

\[
x = x_0 + \alpha m \left(\frac{i x_0}{\alpha}\right)^{(m-1)/m} (t - t_0)
\]

On the downstream boundary \( x = L_0 \) and \( Q = \alpha b_0^m = ix_0 \). Therefore, the relationship between discharge and time for the recession can be written by substitution into equation 26

\[
Q - i L_0 + im x_0^{1/m} Q^{(m-1)/m} (t - t_0) = 0; \quad t_0 < t
\]

Rising hydrographs defined by equation 18 and recession hydrographs defined by equation 27 are shown in Figure 5.

When the duration of rainfall is shorter than the equilibrium time, \( t_e \), a 'partial
equilibrium' hydrograph results. The characteristics of partial equilibrium hydrographs can be readily derived by considering the characteristic shown in Figure 6.

Suppose that rainfall begins at rate $i$ at $t = 0$, and continues until $t = D$. For $t > D$, $i = 0$. Equation 17 is valid along the characteristics beginning at (0,0) described by equation 20 with $C = 0$. At point $a$ and along the line $ac$, $b = iD$. The equation of the rising segment of the hydrograph from $0 < t < D$ is given by equation 18. The outflow rate is a constant in the interval $D < t < b$ because $dh/dt = 0$ along all characteristics when $t > D$. When $t > b$ the recession hydrograph is identical to a recession from equilibrium beginning at $t = D$ because the depth profile along the line $Da$ is an equilibrium profile.

From equation 20 it can be seen that the distance $ac$ is given by

$$ac = L_0 - \alpha i^m D^m$$

(28)

The time interval $cb$ is then given by the following equation:

$$cb = \frac{(iD)^{1-m}}{\alpha m} \left[ L_0 - \alpha i^m D^m \right]$$

(30)

$$= \frac{L_0 (iD)^{1-m}}{\alpha m} - \frac{D}{m}$$

The equation for the partial equilibrium hydrograph for a rain of duration $D < t_e$ can thus be written as:

$$Q = \alpha (it)^m; \quad 0 \leq t \leq D$$

$$Q = \alpha (iD)^m; \quad D \leq t \leq D + \frac{L_0 (iD)^{1-m}}{\alpha m} - \frac{D}{m}$$

$$Q - iL_0 + im\alpha^{1/m} Q^{(m-1)/m} (t - D) = 0; \quad D + \frac{L_0 (iD)^{1-m}}{\alpha m} - \frac{D}{m} < t$$

(31)
Figure 7. Partial equilibrium hydrographs. $K = 5000$, $L_0 = 100$ ft, $i = 1$ inch per hour, $S_0 = 0.05$, $m = 3$

A family of partial equilibrium hydrographs is shown in Figure 7. The flat top is a characteristic of the partial equilibrium hydrograph from a plane. The hydrograph has a sharp peak when $D = t_e$ and again has a flat top when $D > t_e$.

5. Hydraulic resistance in overland flow

Resistance to rainfall-induced overland flow over natural and man-made surfaces may be influenced by several factors. Boundary roughness is frequently much greater than that encountered in ordinary hydraulic structures. At low rates of flow the roughness elements protrude through the free water surface, and at high rates of flow the boundary geometry may change in time and distance because of erosion or because the vegetation is bent over by the flowing water. The impact of raindrops may also have an important effect. On vegetated surfaces the plant leaves and stems may offer more resistance to flow than the soil roughness.

In the past 30 years there have been many laboratory and field investigations aimed at finding the relative importance of the above factors or the appropriate resistance formulae and methods of parameter estimation.

In laboratory studies the most popular approach has been to assume that the Darcy–Weisbach resistance law (equation 10) is appropriate and then to estimate the friction factor, $f$, using measurements of the depth, discharge and slope.

For laminar flow over a smooth surface the theoretical relationship between the friction factor and Reynolds number is

$$f = 24/R_e$$

(32)

For laminar flow over rough surfaces a similar relation has been observed:

$$f = K/R_e; \quad K > 24$$

(33)
where $K$ is a parameter related to the characteristics of the surface and can become very large (40,000) for dense turf. With raindrop impact it appears that the parameter $K$ can be approximated by

$$K = K_0 + Ai^b$$  \hspace{1cm} (34)

where $K_0$ is the parameter without rainfall and $A$ and $b$ are empirical parameters. When $i$ is in inches per hour, the coefficient $A$ is of the order of 10 and the exponent $b$ is approximately unity. Obviously if the surface is smooth ($K_0 = 24$) the raindrop impact effect is important. However, it becomes insignificant for vegetated surfaces.

Transition to an apparently turbulent regime has been reported at Reynolds numbers ranging from 100 to 1000. The higher transition Reynolds numbers, $R_T$, were usually associated with smooth surfaces. The most frequently cited $R_T$ values are in the range $300 < R_T < 500$.

A number of resistance laws have been used for turbulent flow. The Manning formula

$$U = \frac{1.49}{n} S^{1/2} b^{2/3}$$  \hspace{1cm} (35)

has been used most frequently.

Another frequently used relation is the Chézy formula

$$U = C\sqrt{(bs)}$$

$$f = \frac{8g}{C^2}$$  \hspace{1cm} (36)

where $C$ is the Chézy resistance coefficient.

It is not feasible to measure directly the depth of flow over very rough surfaces or in field studies. Therefore, the friction factor must be obtained by inference from the steady state detention storage, from analysis of the rising hydrograph or by optimization techniques utilizing some function of the difference between observed and computed hydrographs for the objective function.

The steady state detention storage can be measured experimentally by integrating the measured recession hydrograph. The analytical expression is obtained by integrating equation 25. The mean detention depth $\bar{b}$, is then given by

$$\bar{b} = \frac{1}{L_0} \int_0^{L_0} b(x) dx = \frac{1}{L_0} \int_0^{L_0} \left( \frac{ix}{\alpha} \right)^{1/m} dx$$

$$\bar{b} = \left( \frac{i}{\alpha} \right)^{1/m} \left( \frac{m}{m + 1} \right) L_0^{1/m}$$  \hspace{1cm} (37)

For laminar flow, $m = 3$ and $\alpha = \frac{8gs}{K\nu}$ where $K$ is the parameter defined in equation 33. Substituting these identities into equation 38 and solving for $K$ we obtain

$$K = \left( \frac{4}{3} \right)^3 \frac{8gs}{\nu L_0 i} \bar{b}^3$$  \hspace{1cm} (39)

where $\bar{b}$ is the measured detention depth.
This method of experimentally finding the value of $K_0$ requires that the flow be laminar for all $x < L_0$. Raindrop impact effect will also be included. The parameters $A$ and $b$ in equation 34 can be estimated by performing experiments with several rainfall intensities.

The exponent $m$ and the parameter $K$ can also be estimated from the rising hydrograph from a plane surface. For laminar flow the equation for the rising hydrograph is

$$Q = \frac{8gS_0}{K
u} i^m t^m$$ (40)

Let

$$Q' = \frac{Q
u}{8gS_0 i^m}$$

then

$$\log Q' = m \log t - \log K$$ (41)

If $Q'$ and $t$ obtained from experimental data are plotted on logarithmic graph paper, $m$ and $K$ can be obtained graphically. Examples of similar plots ($Q$ versus $t$) are shown in Figure 8(a) and (b) for experiments on an asphalt and a turf covered plane. The change in slope of the logarithmic hydrograph may indicate a change in the flow regime for the turf plane whereas it appears that the flow on the asphalt plane was essentially laminar for these runs.

For experimental watersheds or plots that have geometries other than a plane or under natural rainfall the rising hydrograph and the equilibrium detention are very complex; therefore optimization techniques must be utilized for parameter

![Figure 8. Logarithmic hydrographs. (a) Run 64, 59, 56, 52; $s = 0.01$, $i = 3.8$ in/hr; asphalt plane. (b) Run 301, 312, 314, 316; $s = 0.01$, $i = 3.8$ in/hr; turf plane. (After Morgali, 1970)
estimation. In this method the form of the friction law is assumed and starting parameter values are estimated. Equation 8 is then solved numerically and an objective function is evaluated. The parameters are then adjusted by some optimization scheme until the objective function is minimized. One objective function that is frequently used and is appropriate for parameter estimation for single events is

\[ F(K, A, b) = \sum_{i=1}^{n} \left( Q_0(i\Delta t) - Q_c(i\Delta t) \right)^2 \]  

(42)

where the subscripts o and c refer to observed and computed discharge respectively and \( \Delta t \) is a fixed time increment. If a set of \( N \) runoff events is available for analysis the objective function could incorporate the sum of squares of deviation between observed and computed peak rates.

<table>
<thead>
<tr>
<th>Surface</th>
<th>Laminar flow</th>
<th>Turbulent flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete or asphalt</td>
<td>24–108</td>
<td>0.01 –0.13</td>
</tr>
<tr>
<td>Bare sand</td>
<td>30–120</td>
<td>0.01 –0.016</td>
</tr>
<tr>
<td>Gravelled surface</td>
<td>90–400</td>
<td>0.012–0.03</td>
</tr>
<tr>
<td>Bare clay–loam soil (eroded)</td>
<td>100–500</td>
<td>0.012–0.033</td>
</tr>
<tr>
<td>Sparse vegetation</td>
<td>1000–4000</td>
<td>0.053–0.13</td>
</tr>
<tr>
<td>Short grass prairie</td>
<td>3000–10 000</td>
<td>0.10 –0.20</td>
</tr>
<tr>
<td>Bluegrass sod</td>
<td>7000–40 000</td>
<td>0.17 –0.48</td>
</tr>
</tbody>
</table>

Values of the laminar resistance parameter \( K_0 \) and of Manning’s \( n \) and Chézy’s \( C \) for typical surfaces are shown in Table 1. The tabulated values are ranges found in the literature and were obtained by the techniques described above, utilizing data from controlled experiments or from small experimental watersheds. The Chézy \( C \) values were obtained by equating laminar and turbulent friction factors at a transition Reynolds number of 500. Manning’s \( n \) was estimated in the same way for all but the concrete and asphalt surfaces. The slope was taken as 0.05 and the kinematic viscosity was \( 1.2 \times 10^{-5} \). Most of these values were obtained for small areas such as plots or very small watersheds and are undoubtedly biased. One might expect that lower friction factors would be found for larger watersheds with the same vegetative cover because of a greater concentration of runoff in rills and small channels.

Raindrop disturbance parameters \( A \) and \( b \) from Equation 34 found in the literature are shown in Table 2.
Table 2. Rainfall disturbance parameters

<table>
<thead>
<tr>
<th>A</th>
<th>b</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.67</td>
<td>4/3</td>
<td>Izzard (1944)</td>
</tr>
<tr>
<td>27.2</td>
<td>0.4</td>
<td>Li (1972)</td>
</tr>
<tr>
<td>10.0</td>
<td>1.0</td>
<td>Fawkes (1972)</td>
</tr>
</tbody>
</table>

6. Numerical methods for solving kinematic equations for overland flow

Although the kinematic wave equations can be solved analytically unless shocks are present, numerical solutions are more convenient when lateral inflow rates change with time, distance or both. Three finite difference schemes for these equations are shown in Table 3.

All of these methods have been used successfully. The single-step Lax–Wendroff method has second order accuracy and will have the smallest errors of approximation for a given $\Delta x$ and $\Delta t$. The four-point implicit method is unconditionally stable so may save some computation time if larger time steps are used. However, the finite difference equation is non-linear in $h_{j+1}^+\text{d}_j$ and must be solved by an iterative technique. If convergence of this iterative scheme is slow the advantage of unconditional stability may be only apparent.

7. Hydrologic applications

Numerical models based upon the shallow water equations or the kinematic wave approximation can be used to calculate runoff hydrographs resulting from rainfall on urban or agricultural watersheds. The kinematic wave approximation is certainly adequate for overland flow and is preferred to the shallow water equations because of its inherent simplicity. Care must be used if the kinematic approximation is to be used for channel flow because it cannot account for backwater effects. Either model has advantages over the unit hydrograph approach for predicting runoff for ungauged watersheds because the model structure and resistance parameters can be estimated without prior rainfall and runoff records.

The first step in applying these models is to decide upon the model geometry. Two approaches have been used. The first attempts to maintain a close geometric similarity between the prototype watershed and the idealized network or cascade of planes and channels that specify the model geometry (Harley, Perkins and Eagleson, 1970). The second approach represents all watersheds, regardless of their complexity, by simple geometric elements such as a combination of two planes and a channel or a linearly converging section (Wooding, 1966).

An example of the geometric simplification involved in the first approach is shown in Figure 9. In this example a parking lot is represented by six overland flow planes that contribute lateral inflow to three swale (channel) flow elements. If the kinematic wave equations are used, the hydrographs for each of the planes can be computed first, then the hydrographs at the downstream boundary of each channel could be computed with the time varying outflow of the planes treated as lateral
<table>
<thead>
<tr>
<th>Method</th>
<th>Finite difference equation</th>
<th>Order of approximation</th>
<th>Linear stability criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-step Lax—Wendroff</td>
<td>$b_{j}^{i+1} = b_{j}^{i} - \alpha \Delta t \left[ \frac{(b_{j+1}^{m} - b_{j-1}^{m})}{2 \Delta x} - \frac{1}{2} \alpha (q_{j+1}^{i} + q_{j-1}^{i}) \right]$</td>
<td>$O(\Delta x)$</td>
<td>$\frac{\Delta t}{\Delta x} \leq \frac{1}{amh^{m-1}}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{\Delta t^{2} m}{4 \Delta x} \left[ (b_{j+1}^{m-1} + b_{j}^{m-1}) \frac{(b_{j+1}^{m} - b_{j}^{m})}{\Delta x} - \frac{1}{2} (q_{j+1}^{i} + q_{j-1}^{i}) \right]$</td>
<td>$0(\Delta x)^{2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$- (b_{j}^{m-1} + b_{j-1}^{m-1}) \left[ \frac{(b_{j}^{m} - b_{j}^{m-1})}{\Delta x} - \frac{1}{2} (q_{j}^{i} + q_{j-1}^{i}) \right]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$+ \frac{2 \Delta x}{am \Delta t} (q_{j+1}^{i} - q_{j}^{i})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Upstream differencing</td>
<td>$b_{j}^{i+1} = b_{j}^{i} + m \alpha \frac{\Delta t}{\Delta x} (b_{j}^{m} - b_{j-1}^{m}) + q_{j}^{i} \Delta t$</td>
<td>$0(\Delta x)$</td>
<td>$\frac{\Delta t}{\Delta x} \leq \frac{1}{2.75amh^{m-1}}$</td>
</tr>
<tr>
<td>Brakensieck's four-point implicit</td>
<td>$\frac{b_{j}^{i+1} - b_{j}^{i} + b_{j+1}^{i+1} - b_{j-1}^{i+1}}{2 \Delta t} + \alpha (b_{j}^{i+1m} - b_{j-1}^{i+1m})$</td>
<td>$0(\Delta x)$</td>
<td>Unconditionally stable</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{\Delta t}{\Delta x} (b_{j}^{i+1m} - b_{j-1}^{i+1m})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$- 1/(q_{j}^{i+1} + q_{j}^{i} + q_{j}^{i} + q_{j}^{i+1}) = 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Adapted from Kibler and Woolhiser, 1970.*
Figure 9. Geometric characteristics of small urban area (after Schaake, 1965)
inflow to the channel and with the discharge from the upstream channel section as the upstream boundary condition. Appropriate resistance parameters can be selected from Table 1. Hydrographs obtained by an explicit numerical solution of the kinematic wave equations are shown in Figure 10. The friction parameters were chosen as the extremes for paved surfaces shown in Table 1. The transition Reynolds number, $R_T$, was 500. The observed peak discharge lies between the discharges computed for the extreme values of the friction factor. Schaake obtained his hydrograph by solving the 'complete' equations numerically. The close agreement between Schaak's solution and the kinematic hydrograph reinforces the conclusion that the kinematic model is adequate for overland flow.

An example of the representation of an agricultural watershed by cascades of planes and channels is shown in Figure 11.

Simple kinematic models of overland flow are now being used in hydrologic practice. These models will be especially useful in design situations where they can
be used to determine the effect of detention storage or drainage design on peak discharge rates from urban watersheds. In rural areas they will be useful in evaluating the hydrologic effects of terraces, diversions and detention structures and in predicting peak runoff rates where no data are available for identifying simpler models or for developing regionalized information.