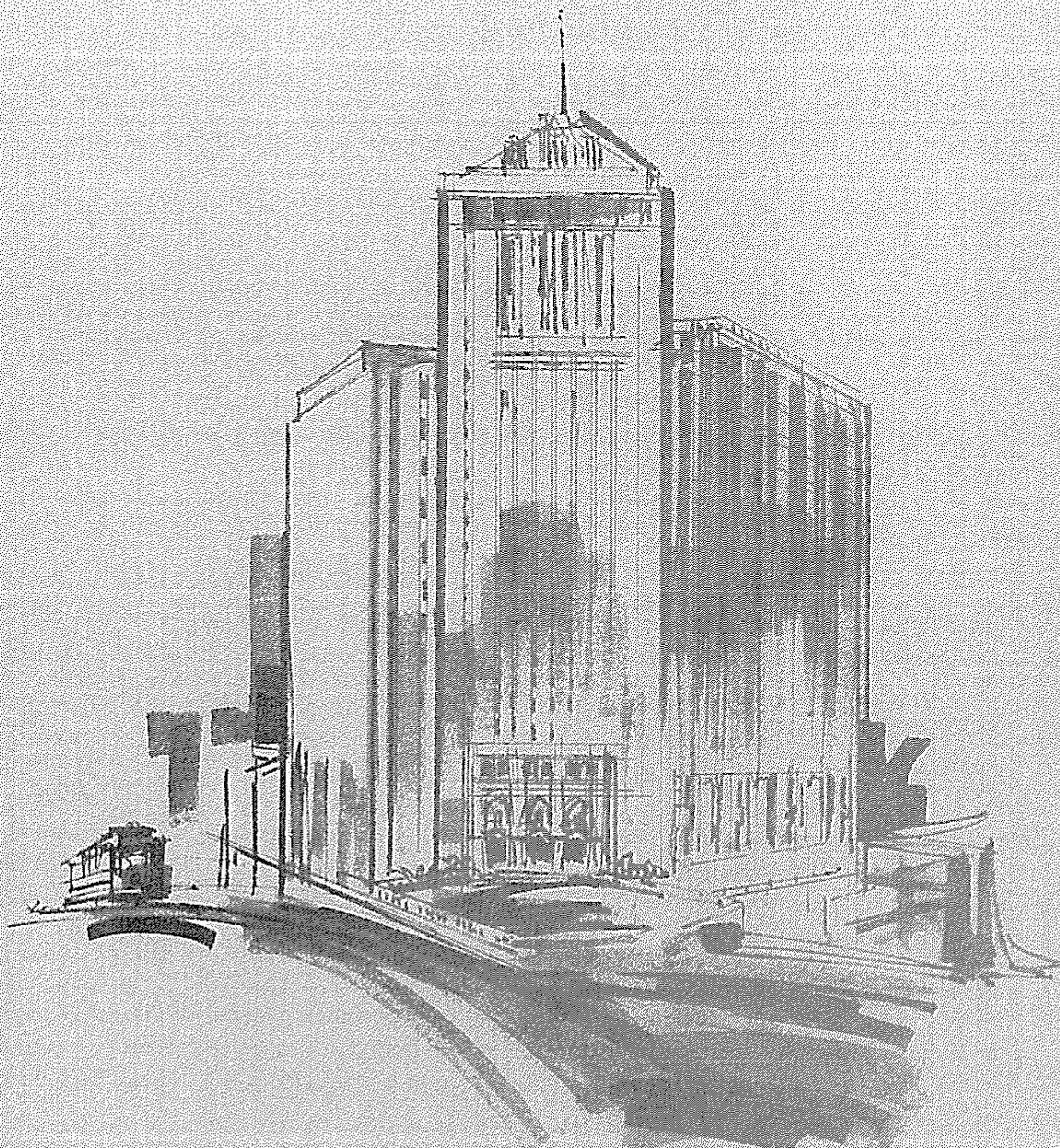

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THE USE OF THE SHALLOW WATER EQUATIONS IN RUNOFF COMPUTATION

by

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Introduction

In the past few years the shallow water equations have become popular tools for use in research on surface runoff problems. Many mathematically minded engineers have been intrigued by the possibility of integrating these equations to obtain rational hydrographs. Implicit in the use of these equations is the objective of synthesizing the surface runoff component of a watershed as a distributed, nonlinear system. Although the shallow water equations are perhaps the most complex formulation of the physics of the problem that can be handled conveniently, the many idealizations necessary in their derivation prevent the solutions from conforming to reality. On the other hand, the equations indicate important parameters and the general form of the solution.

The basic component of mathematical models describing surface runoff is an overland flow plane discharging into a channel as shown in Figure 1. Unsteady, spatially varied flow over the plane surface is described by the shallow water equations:

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = q \tag{1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = g(S_o - S_f) - \frac{qu}{h} \tag{2}$$

where h = depth (dimensions of length, L),

u = velocity (L/T),

t = time (dimensions of time, T),

x = horizontal distance (L),

q = lateral inflow (L^3/TL^2),

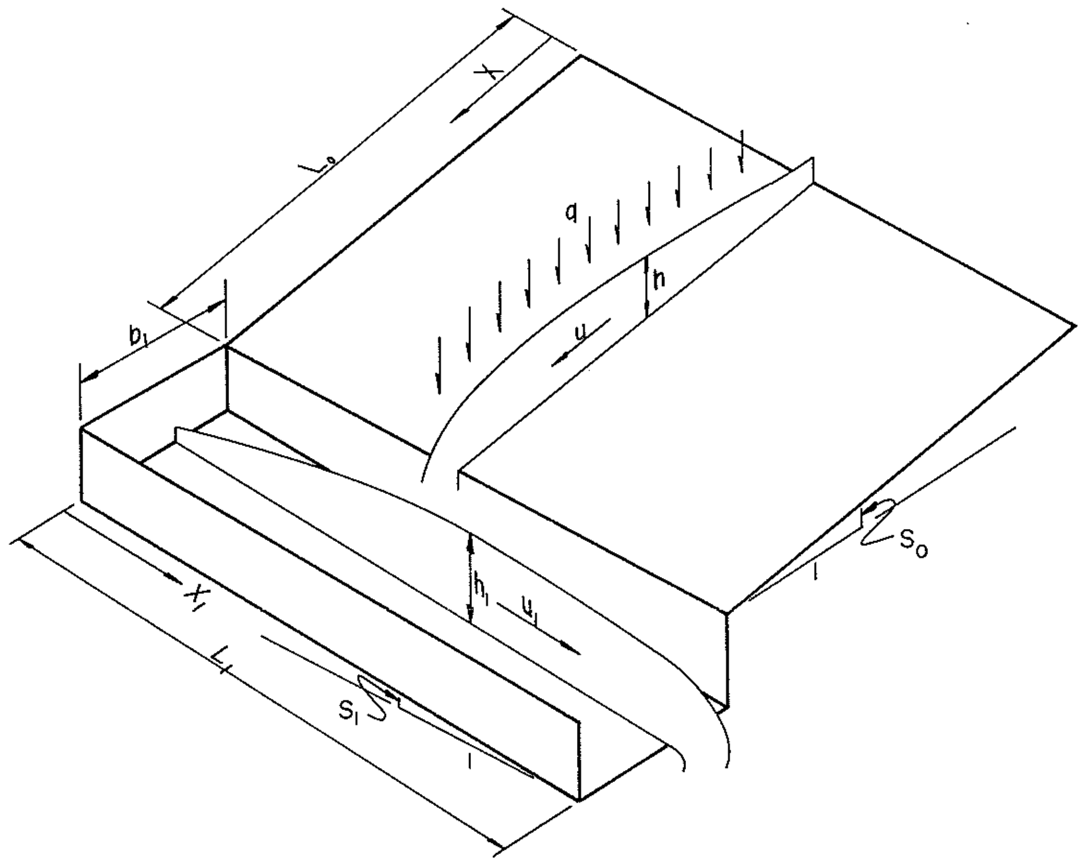

FIG. I OVERLAND FLOW PLANE DISCHARGING INTO A WIDE RECTANGULAR CHANNEL

g = acceleration of gravity (L/T^2),

S_o = the channel or plane slope (L/L), and

S_f = the "friction slope" defined (in this paper) by the Chezy equation $S_f = \frac{u^2}{C^2 h}$ where C is an empirical constant. The Manning equation could also be used.

The system forms two simultaneous equations in the dependent variables, h and u, as functions of the independent variables, x and t. Because the equations are nonlinear, the solutions usually must be carried out on a digital computer.

A similar form of the shallow water equations exists for flow in channels. These equations can be made as complex as desired within the general framework of shallow water theory. That is, the plane or channel may be warped; the lateral inflow may be a function of space and time and can also be varied to account for infiltration; several types of boundary conditions are suitable; and the friction relationship can take a number of different forms. However, all of these variations cannot be made to account for the many natural variations in a drainage area in any practical way. On the other hand, the shallow water equations do account for the physics of the surface runoff process much better than any of the previous methods, either theoretical or empirical.

The Runoff Process

It is obvious even to the casual observer that continuous sheet flow, which is assumed as the basis for derivation of the shallow water equations, does not occur in nature. On natural surfaces, water forms rivulets which join in a branching pattern. Obstacles such as grass, pebbles, and natural debris significantly disturb the flow. Even when the surface is paved the flow is discontinuous, the discontinuities being caused by a friction-slope instability or by surface tension. In the early stages of channel flow many of the same anomalies are present. Only after an appreciable flow has collected in a channel does it appear that the assumptions upon which the derivation is based are valid.

The definition of the frictional resistance to flow is an extremely difficult problem. The flow regime may vary from laminar to turbulent as a function of distance and time depending on variations in depth of flow, geometry of the roughness elements and disturbance by raindrop impact. Manning's or Chezy's empirical equations have commonly been used to define frictional resistance, usually with the recognition that their application is rather tenuous in the case of overland flow. Indeed it might be more realistic to describe the relation between velocity and depth by a parametric model of the form cited by Wooding [4].

$$u = \alpha h^{n-1}$$

where α and n are merely parameters to be optimized to obtain the best fit of computed and observed hydrographs according to some specified criterion. These optimized parameters could then be correlated with physical properties of the surface. A more detailed approach such as that of Harbaugh and Chow [2] results in an increase in the number of parameters with the associated problem of physical interpretation and estimation.

The Definition of Significant Parameters

It is useful to write the shallow water equations in dimensionless form. Dimensionless variables are defined by dividing each variable by physically significant normalizing quantities having the same dimensions:

$$x_* = \frac{x}{L_o} \; ; \; t_* = \frac{tV_o}{L_o} \; ; \; u_* = \frac{u}{V_o} \; ; \; h_* = \frac{h}{H_o} \; ; \; q_* = \frac{qL_o}{V_oH_o}$$

Using these definitions in equations (1) and (2) there results [3,6]:

$$\frac{\partial h_*}{\partial t_*} + u_* \frac{\partial h_*}{\partial x_*} + h_* \frac{\partial u_*}{\partial x_*} = q_* \tag{3}$$

$$\frac{\partial u_*}{\partial t_*} + u_* \frac{\partial u_*}{\partial x_*} + \frac{1}{F_o^2} \frac{\partial h_*}{\partial x_*} = k(1 - \frac{u^2_*}{h_*}) - q_I \frac{u_*}{h_*} \tag{4}$$

The two dimensionless parameters appearing in these equations are:

$$F_o = \frac{V_o}{\sqrt{gH_o}} \qquad \text{and} \qquad k = \frac{S_o L_o}{F^2_o H_o}$$

F_o is a Froude number based upon the reference quantities. The other number, k (called the kinematic flow number in this paper),[1] indicates the importance of slope and friction and is very significant in the solution of the equations.

Solutions to equations (3) and (4) with a step function lateral inflow are shown in Figure 2 for an F_o of 1 and a variable k. As k becomes large, the dynamic solution approaches the kinematic wave solution [6]. Under these conditions slope and friction dominate the dynamic effect and as $k \to \infty$ equation (4) reduces to

$$u_*^2 = h_* \tag{5}$$

The solution to the simultaneous equations (3) and (5) can be obtained analytically for a step function lateral inflow [4,6]

$$Q_* = u_* h_* = t_*^{3/2} \; ; \; 0 < t_* \leq 1 \tag{6}$$

$$Q_* = 1 \quad ; \; 1 < t_*$$

This solution is independent of the parameters F_o and k.

An examination of Figure 2 shows that when k > 20 the kinematic wave approximation is very good indeed. For many overland flow problems the kinematic flow number is well over 1,000 in which case the solutions would be practically indistinguishable. This is fortunate because as k becomes large the dynamic equation is poorly posed and numerical solution becomes very difficult. Although numerical solution of the kinematic wave equations is not trivial, it is not as difficult as the solution of the complete equations and, furthermore, analytic solutions are attainable in many cases.

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[1]Dr. D. L. Brakensiek at the Annual Meeting of the American Geophysical Union, April 1967, suggested the term "kinematic wave number." Because the phrase "wave number" has a quite different connotation, the writers prefer the indicated terminology.

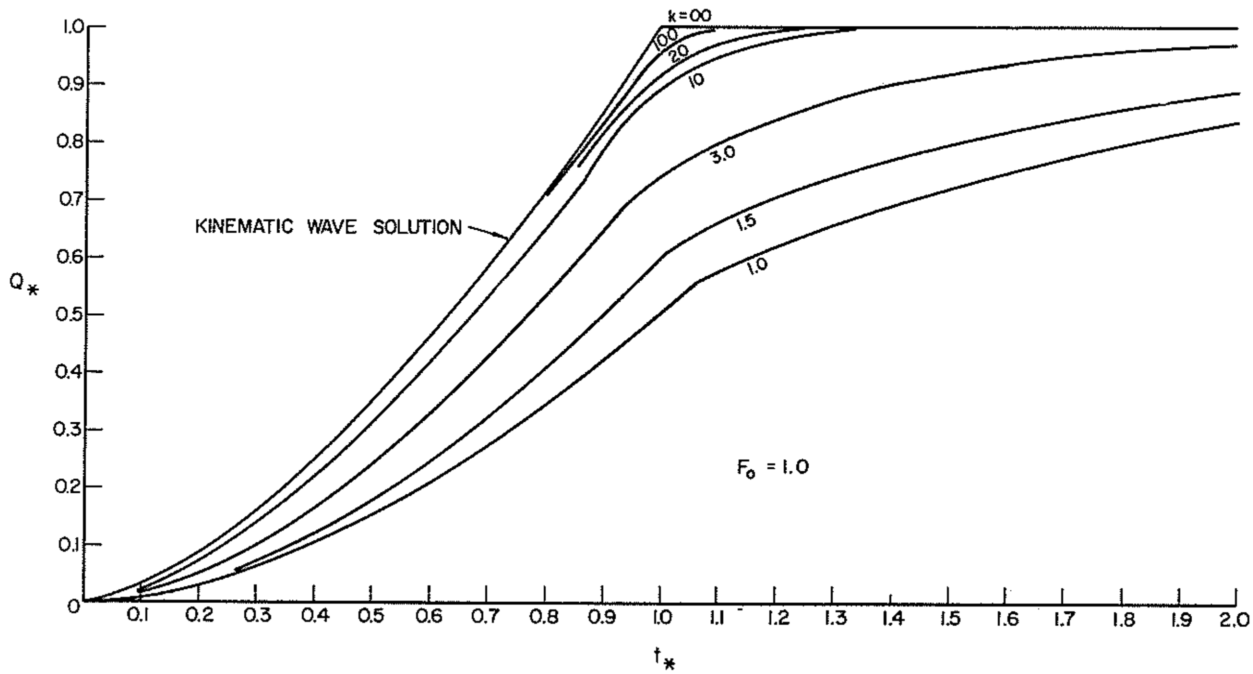

FIG. 2 THE RISING HYDROGRAPH - VARIATION WITH k

Equation (6) represents the dimensionless rising hydrograph for kinematic flow without reference to the particular problem, given that the lateral inflow can be represented by a step function. Similar solutions can be obtained for other input patterns such as single pulses of varying lengths or a series of pulses of varying length or intensity. Because the problem is nonlinear, superposition cannot be used and the response to any particular input *pattern* is unique. In a recent paper, J. C. I. Dooge [1] states that the kinematic wave formulation appears to be a special case of what he has called uniformly nonlinear models--a class of nonlinear models consisting of a series of nonlinear elements all of which are nonlinear to the same degree. The implications of this observation are not presently apparent and this appears to be a good area for further research.

The equations describing flow per unit width in a wide rectangular channel receiving inflow from a plane are identical to equations (3) and (4) if the dimensionless variables are defined as follows:

$$x_{**} = \frac{x_1}{L_1} \; ; \; u_{**} = \frac{u_1}{V_1} \; ; \; h_{**} = \frac{h_1}{H_1} \; ; \; q_{**} = \frac{q_1 b_1}{q_o L_o}$$

$$\text{and} \quad \frac{t\ V_1}{L_1} = \frac{t\ V_o}{\lambda\ L_o}$$

where $\lambda = \frac{V_o L_1}{L_o V_1}$ is the ratio of the characteristic times. The other normalizing quantities, the Froude number and the kinematic flow number, are as prevously defined except they refer to the channel rather than to the plane. The parameter λ is a time scale and can be computed for any particular case in terms of the geometry and roughness as

$$\lambda = \frac{L_1}{L_o}\left(\frac{C_o^2\ b_1\ S_o}{C^2{}_1\ L_1\ S_1}\right)^{1/3}$$

where C_o and C_1 are the Chezy coefficients for the plane and the channel, respectively. This definition of a time scale corresponds very closely to that used by Wooding [4] and has the same significance. A small value of λ indicates a very short channel which will have very little effect on the hydrograph. As λ becomes larger, the channel effects become more important.

If one considers the case where the lateral inflow to a channel such as shown in Figure 1 is the kinematic response of a plane to a step input, equation (3) becomes

$$\frac{\partial\ h_{**}}{\partial\ t_{**}} + u_{**}\frac{\partial\ h_{**}}{\partial\ x_{**}} + h_{**}\frac{\partial\ u_{**}}{\partial\ x_{**}} = (\lambda\ t_{**})^{3/2}\ ;\ 0 < t_{**} < 1/\lambda \tag{7}$$

$$= 1\ ;\ 1/\lambda \leq t_{**}$$

when the expression $u_{**}{}^2 = h_{**}$ is substituted into (7) we obtain

$$\frac{\partial\ h_{**}}{\partial\ t_{**}} + \frac{3}{2}\ h_{**}{}^{1/2}\ \frac{\partial\ h_{**}}{\partial\ x_{**}} = (\lambda\ t_{**})^{3/2} \tag{8}$$

which can be reduced to the ordinary differential equation

$$\frac{d\ h_{**}}{d\ t_{**}} = (\lambda\ t_{**})^{3/2}\ ;\ 0 < t_{**} < 1/\lambda \tag{9}$$

$$= 1 \qquad ;\ 1/\lambda < t_{**}$$

in the characteristic direction defined by

$$\frac{dx_{**}}{dt_{**}} = \frac{3}{2}\ h_{**}{}^{1/2} \tag{10}$$

The solution plane for this problem is shown in Figure 3. The expression for the rising hydrograph can be obtained analytically for $0 \leq t_{**} < t_2$

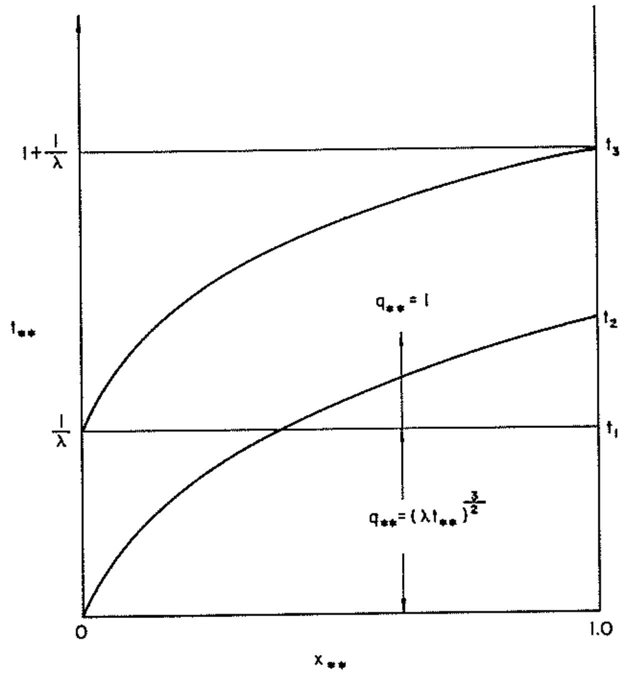

FIGURE 3 SOLUTION PLANE – CHANNEL WITH KINEMATIC LATERAL INFLOW

$$Q_{**} = u_{**}h_{**} = \lambda^{9/4} \left(\frac{2}{5}\right)^{3/2} t_{**}^{15/4}; \; 0 < t_{**} < 1/\lambda \tag{11}$$

$$Q_{**} = \left(t_{**} - \frac{3}{5\lambda}\right)^{3/2} \; ; \; 1/\lambda < t_{**} < t_2 \tag{12}$$

where

$$t_2 = \frac{3}{5\lambda} + \left[1 - \left(\frac{2}{5}\right)^{1/2} \left(\frac{4}{15}\right) \left(\frac{1}{\lambda}\right)^{3/2}\right]^{2/3}$$

Equations (9) and (10) cannot be integrated analytically from a point on the upstream boundary $(0,t_o)$ where $t_o < 1/\lambda$ (Figure 3) so the discharge over the interval $t_2 < t_{**} < t_3$ must be obtained by numerical integration. (Wooding [4,5] obtained analytical expressions for all segments because he used the linear friction relationship $u = \alpha h$ for overland flow.)

The kinematic rising hydrograph for a channel receiving kinematic lateral inflow from a plane with a step-function input is shown for one value of the parameter λ in Figure 4. Also in Figure 4 the kinematic solution is compared with the dynamic solution for the parameter values $F_o = 1$, $k = 5$ computed by the method of characteristics. As the time ratio, λ, becomes large in such dimensionless plots the lateral inflow curve will approach a step input and the effects of the plane will be negligible.

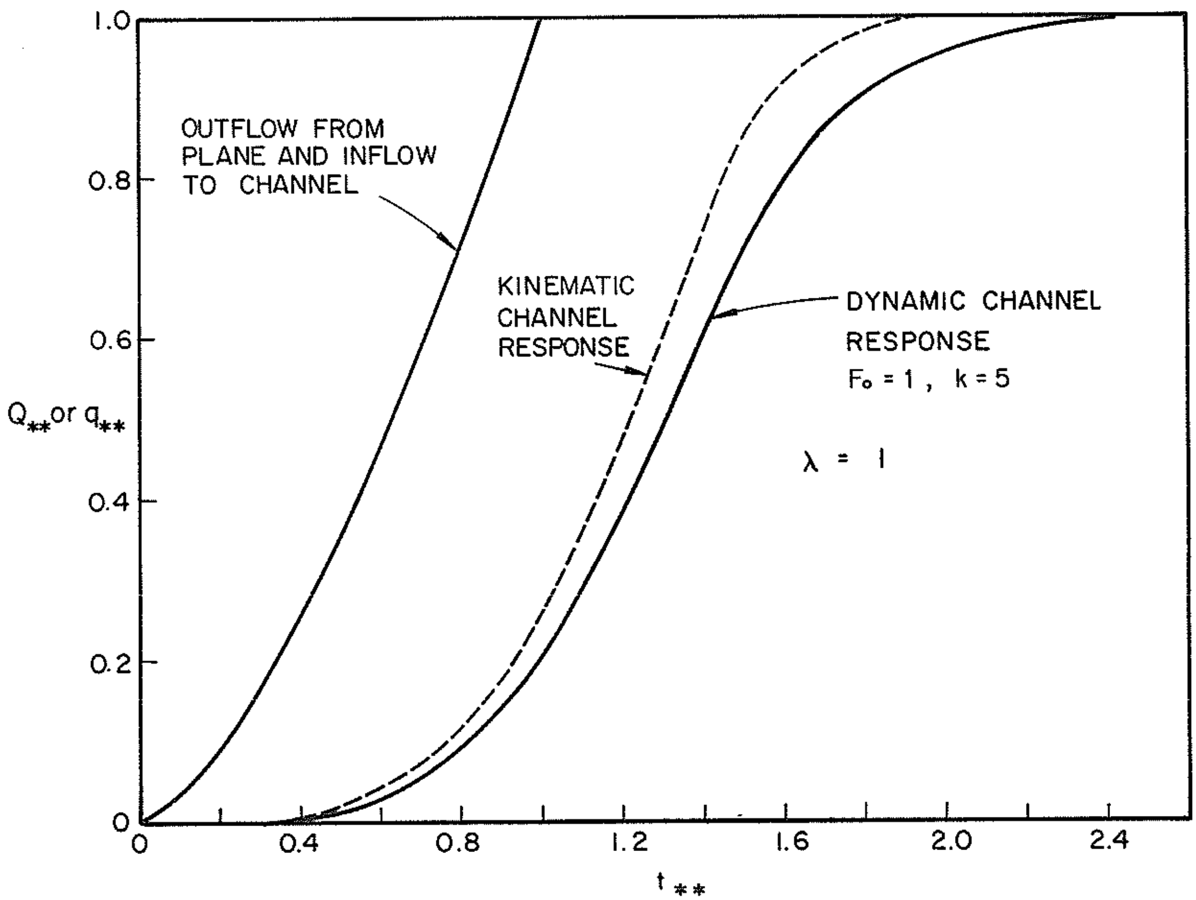

FIGURE 4 DYNAMIC RISING HYDROGRAPH

The rerouting of overland flow through a channel as illustrated in Figure 4 has some important implications in the interpretation of experimental overland flow data. These data commonly show an inflection point on the rising hydrograph and there is often a considerable time lag before measurable outflow occurs. These data do not necessarily indicate that the kinematic approximation is not a good one, because both effects can be obtained by rerouting a kinematic outflow from a plane through a short reach of channel as shown in Figure 4. Such rerouting is a physical fact in experiments where the outflow is concentrated for measuring purposes.

Possibilities for Further Research

Research into the application of the shallow water equations in hydrology must follow certain logical steps if the physical significance of the equations is not to be compromised by gross oversimplification of watershed geometry. Now that it appears that the kinematic wave equation is quite appropriate for most cases involving overland flow, a comprehensive reanalysis of experimental data on overland flow should be carried out to see if the kinematic wave formulation is an improvement over the simple nonlinear storage models. Several other questions are unresolved with regard to overland flow. Can a concave or a convex slope be represented by a plane? If so, can an objective technique for estimating the slope of the plane be developed? Are the methods for estimating parameters from experimental data sensitive to errors in the input or output? It would seem that such questions must be answered *before* one attempts to simulate any watershed that departs very much from a plane.

The important question of what frictional laws are valid still remains open. Indeed, a great amount of fine detail of the watershed can probably be lumped into the frictional parameters. For kinematic flow such research might be immediately productive since the equations indicate that the friction coefficient (for a given relationship) alters the time base of the hydrograph but does not affect the flow ordinates.

It is apparent that many of the above questions are related to the problem of determining an optimum model structure. This is the dilemma faced in system synthesis: One wishes to select the components of the system in such a manner that they retain physical significance; yet, the natural complexity of most watersheds is such that many of the details will inevitably be omitted in the model. The key question is how much simplification can be made while still retaining physical significance?

Conclusions

The runoff process is extremely complex and cannot be described mathematically now or in the foreseeable future. The shallow water equations provide the closest description available, but the solution to these nonlinear equations appears to be too complex to gain general popularity, at least as far as the small drainage area is concerned. (The river problem seems to be easier to describe by data input to standard programs.) However, solutions to the shallow water equations provide insight into the physical process and aid in evaluating less complex models.

The shallow water equations have indicated the important parameters in the overland flow problem. They indicate the degree of approximation in the easier kinematic wave solution. In a like manner they can indicate the strengths and weaknesses of any other method. Work is now proceeding at Cornell University to evaluate some of the commonly used methods, such as the unit hydrograph method, in terms of solutions of the shallow water equations. Indeed, such solutions can often be used to provide "clean data" with which the less sophisticated methods are evaluated, thus abridging the tremendous difficulties in obtaining good data. Answers can be obtained to such important questions as: "For what values of the parameters is the process reasonably linear or uniformly nonlinear?"

Many aspects of the runoff process are stochastic. For example the exact stream or rivulet pattern can probably never be described deterministically. Also because the input to a hydrologic system (rainfall) is stochastic, the output (streamflow) can only be described in terms of its probability laws. However, it is reasonable to separate and study those parts of the process that are deterministic for these are the components subject to control by engineering works or agricultural practices.

There will undoubtedly be some improvement in runoff predictions made by the shallow water equations. However, the writers feel that further research into the *direct* use of the shallow water equations, as opposed to using the equations to obtain a better understanding of the process, will lead to only small improvements.

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