

APPROXIMATE FORM OF GREEN-AMPT INFILTRATION EQUATION

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ABSTRACT: We develop an approximation of the Green-Ampt infiltration equation by using the first two terms of a Taylor-series expansion of the equation. The resulting approximate equation is in the form of the Philip's equation, with an added term to account for the error in the approximation. The Taylor-series approximation is compared with Li et al.'s 1976 quadratic approximation for the case of constant rainfall with variable time to ponding. The maximum error for the new approximation is about 3.5%; and the maximum error for the quadratic approximation is about 8%. For a range of values of the ratio of infiltrated depth to capillary potential of 0.1 to 150 the approximation of the Taylor series fits the Green-Ampt infiltrated depth more closely than the quadratic approximation. It is shown that the approximation of the Taylor series gives less error than the quadratic approximation for coarser textured soils.

INTRODUCTION

The Green-Ampt equation is becoming more widely used, partly as a result of the Water Erosion Prediction Project (WEPP) ("USDA" 1989). In addition to the fact that the parameters in the equation have physical significance, experimental work has been completed or is underway to obtain values for the parameters based on soil texture and on the effects of management (Rawls et al. 1982, 1989). In general, solutions for the Green-Ampt equation use a Newton-Raphson method to obtain the infiltrated depth at a given time. Because of the iterative form of the Newton-Raphson method, the solution in practice is implemented on programmable calculators or more generally as a computer program. The disadvantage of the Newton-Raphson solution is that the total infiltration depth and final rate cannot be obtained in one step, but as a summation of sequential solutions of the Newton-Raphson iteration. It is convenient in some instances, however, to have a solution to Green-Ampt which avoids the Newton-Raphson method. For the case of initial ponded conditions, Li et al. (1976) developed a quadratic approximation of the Green-Ampt equation. In this paper, we use the nondimensional variables of Li et al. and develop a form of the Green-Ampt based on a Taylor series expansion which approximates the form of the Philip's (1957) infiltration equation.

GREEN-AMPT EQUATION

The rate form of the Green-Ampt equation for the one-stage case of initially ponded conditions, assuming the ponded water depth is shallow, is

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$$f(t) = K_e \left[1 + \frac{\psi\theta_d}{F(t)} \right] \dots\dots\dots (1)$$

where $f(t) = dF(t)/dt$ = infiltration capacity (L/T); K_e = effective saturated conductivity (L/T); ψ = average capillary exponential (L); θ_d = moisture deficit (L/L); and $F(t)$ = cumulative infiltrated depth (L). The soil-moisture deficit can be computed as

$$\theta_d = \eta_e - \theta_i \dots\dots\dots (2)$$

where η_e = effective porosity (L/L); and θ_i = initial volumetric water content (L/L). Eq. (1) is a differential equation, which is solved as

$$K_e t = F(t) - \psi\theta_d \ln \left[1 + \frac{F(t)}{\psi\theta_d} \right] \dots\dots\dots (3)$$

where t = time (T). Eq. (3) can be solved for infiltrated depth for successive increments of time using a Newton-Raphson iteration and the solution used in (1) to obtain the instantaneous infiltration rate.

For the case of constant rainfall, infiltration is a two-stage process; the first stage occurs when the rainfall is less than the potential rate and the second occurs when the rainfall rate is greater than the potential infiltration rate. Mein and Larson (1973) modified the Green-Ampt equation for this two-stage case by computing a time to ponding, t_p (T), as

$$t_p = \frac{K_e \psi \theta_d}{i(i - K_e)} \dots\dots\dots (4)$$

where i = rainfall rate (L/T). After time to ponding, the time in (3) is corrected using the time to ponding and the time, t_s (T), that the potential infiltration rate equals the rainfall rate. The time correction, t_c (T), is computed as

$$t_c = t + t_s - t_p \dots\dots\dots (5)$$

where t_s is computed as

$$t_s = \frac{F_{tp} - \psi\theta_d \ln \left(1 + \frac{F_{tp}}{\psi\theta_d} \right)}{K_e} \dots\dots\dots (6)$$

where F_{tp} = amount of cumulative infiltration (L) at the time to ponding, which, for the case of constant rainfall, is equal to the cumulative rainfall depth at the time to ponding. The corrected time as computed by (5) is used in (3), which is solved as with the one-stage case.

It is for the foregoing two cases (initial ponded conditions and two-stage infiltration under constant rainfall) that we develop approximate equations that avoid the Newton-Raphson iteration in the solution.

Li et al.'s Approximation for Initial Ponded Conditions

Li et al. (1976) used the nondimensional terms

$$t_* = \frac{tK_e}{\psi\theta_d} \dots\dots\dots (7)$$

$$F_*(t_*) = \frac{F(t)}{\psi\theta_d} \dots\dots\dots (8)$$

to write (3) as

$$t_* = F_*(t_*) - \ln[1 + F_*(t_*)] \dots\dots\dots (9)$$

By using the first term of a power-series expansion of the natural log term in (9), Li et al. (1976) derived the following quadratic approximation of infiltrated depth, $F_{q*}(t_*)$, for the case of initial ponded conditions as

$$F_{q*}(t_*) = \frac{1}{2} (t_* + \sqrt{t_*^2 + 8t_*}) \dots\dots\dots (10)$$

If the infiltration rate is nondimensionalized as

$$f_*(t_*) = \frac{f(t)}{K_e} \dots\dots\dots (11)$$

then Li et al.'s infiltration rate, $f_{q*}(t_*)$, is computed as

$$f_{q*}(t_*) = 1 + \frac{1}{F_{q*}(t_*)} \dots\dots\dots (12)$$

Green-Ampt Written in Form of Philip's Equation

We now develop an alternative approximation of the Green-Ampt equation by rewriting it in the form of the Philip's equation. By rearranging (9) and squaring both sides, we get a function, g

$$g(t_*) = [F_*(t_*) - t_*]^2 = \ln^2[1 + F_*(t_*)] \dots\dots\dots (13)$$

A Taylor-series expansion of $g(t_*)$ using the first two terms of the expansion is

$$g(t_*) \cong g(0) + g'(0)t_* \dots\dots\dots (14)$$

where g' refers to the first derivative of g with respect to t_* . The first term in the expansion is zero because $F_*(t_* = 0) = 0$. The second term can be evaluated by taking the derivative with respect to t_* and evaluating it at zero as

$$g' = 2 \ln[1 + F_*(t_*)] \frac{f_*(t_*)}{1 + F_*(t_*)} = \frac{2 \ln[1 + F_*(t_*)]}{F_*(t_*)} \dots\dots\dots (15)$$

Because the numerator and denominator are equal to zero at $t_* = F_*(t_*) = 0$, we can use L'Hôpital's rule to evaluate the limit as $F_*(t_*) \rightarrow 0$

$$g'(0) = \lim_{F_*(t_*) \rightarrow 0} \frac{d\{2 \ln[1 + F_*(t_*)]\}}{dF_*(t_*)} \dots\dots\dots (16a)$$

$$g'(0) = \lim_{F_*(t_*) \rightarrow 0} \frac{2}{1 + F_*(t_*)} \dots\dots\dots (16b)$$

$$g'(0) = 2 \dots\dots\dots (16c)$$

Then (14) becomes

$$g(t_*) \cong 2t_* \dots\dots\dots (17)$$

and by substituting into (13) we get

$$[F_*(t_*) - t_*]^2 \cong 2t_* \dots\dots\dots (18)$$

Because $F_*(t_*) \geq t_*$, (18) becomes

$$F_*(t_*) \cong t_* + (2t_*)^{1/2} \dots\dots\dots (19)$$

which has a form similar to the Philip's equation. If we include an error term, ϵ_* , in (19) to account for the approximation, we get the nondimensional equation for infiltrated depth as

$$F_*(t_*) = t_* + (2t_*)^{1/2} + \epsilon_* \dots\dots\dots (20)$$

The error term can be approximated by a power term computed by a non-linear least-squares fitting of the coefficients with the difference of $F(t)/(\psi\theta_d)$ as computed by (3) and $F_*(t_*)$ as computed by the first two terms of the right-hand side of (20). The goodness of fit for the power term was an $r^2 = 0.96$ with a standard error of 0.016 $F_*(t_*)$ units for a range of $F_*(t_*)$ between 0 and 20. The resulting approximation of the Taylor series for cumulative infiltrated depth, $F_{p*}(t_*)$, is

$$F_{p*}(t_*) = t_* + (2t_*)^{1/2} - 0.2987t_*^{0.7913} \dots\dots\dots (21)$$

The equation for infiltration rate, $f_{p*}(t_*)$ is

$$f_{p*}(t_*) = 1 + \frac{1}{F_{p*}(t_*)} \dots\dots\dots (22)$$

Eqs. (10) and (22) are for the case of initially ponded conditions. To write (10) and (21) for the case of two-stage infiltration with constant rainfall, we nondimensionalize the corrected time, t_{c*} , as

$$t_{c*} = \frac{K_c(t + t_s - t_p)}{\psi\theta_d} \dots\dots\dots (23)$$

Eqs. (10) and (20) are then

$$F_{q*}(t_{c*}) = \frac{1}{2} \left[t_{c*} + \left(t_{c*}^2 + 8t_{c*} \right)^{1/2} \right] \dots\dots\dots (24)$$

$$F_{p*}(t_{c*}) = t_{c*} + (2t_{c*})^{1/2} - 0.2987t_{c*}^{0.7913} \dots\dots\dots (25)$$

EXAMPLE

To illustrate the behavior of the two equations in reproducing Green-Ampt infiltration, we choose the Green-Ampt parameter values for K_c , ψ , and η_c from Rawls et al. (1982) for a sandy loam, silt, and silty clay loam texture soils. The parameter values with the associated t_{c*} computed from (23) for a 60-mm/hr rainfall of 1-hr duration are listed in Table 1. The effective conductivity term is for a bare, uncrusted soil surface condition.

TABLE 1. Green-Ampt Parameter Values from Rawls et al. (1982) and Nondimensional Corrected Time for 60-mm/h Rainfall of 1-h Duration

Soil texture (1)	K_e (mm/h) (2)	ψ (mm) (3)	η_e (mm/mm) (4)	θ_i (mm/mm) (5)	t_{c*} (6)
sandy loam	22	90	0.41	0.50	1.11
silt	5	190	0.42	0.50	0.12
silty clay loam	1.8	253	0.43	0.20	0.03

An example calculation for the silt soil for total infiltration and final infiltration rate is given in the following.

To calculate the total infiltration depth and final rate, first compute the time to ponding, t_p , using (4)

$$t_p = \frac{(5)(190)(1 - 0.5)(0.42)}{(60)(60 - 5)} = 0.060 \text{ hr} \dots\dots\dots (26)$$

The cumulative infiltration depth, which occurs before the time to ponding, is

$$F_p = i t_p = (60)(0.060) = 3.60 \text{ mm} \dots\dots\dots (27)$$

The time shift, t_s , is computed using (6)

$$t_s = \frac{(3.60) - (190)(1 - 0.5)(0.42) \ln \left(1 + \frac{(3.60)}{(1 - 0.5)(190)(0.42)} \right)}{(5)}$$

$$= 0.031 \text{ hr} \dots\dots\dots (28)$$

The time correction, t_c , is then

$$t_c = t + 0.031 - 0.060 = t - 0.029 \text{ hr} \dots\dots\dots (29)$$

For a duration of 1 hr, the nondimensional corrected time, t_{c*} , is

$$t_{c*} = \frac{[(1) - (0.029)](5)}{(190)(1 - 0.5)(0.42)} = 0.122 \dots\dots\dots (30)$$

The nondimensional and redimensionalized infiltrated depths from (24) and (8) are

$$F_{q*}(0.122) = \frac{1}{2} \{[(0.122) + [(0.122)^2 + (8)(0.122)]^{1/2}]\} = 0.559 \dots\dots (31)$$

$$F_q = (0.559)(39.9) = 22.3 \text{ mm} \dots\dots\dots (32)$$

and for nondimensionalized and redimensionalized final rate from (12) and (11) are

$$f_{q*}(0.122) = 1 + \frac{1}{0.559} = 2.79 \dots\dots\dots (33)$$

$$f_q = (2.79)(5) = 13.9 \text{ mm} \dots\dots\dots (34)$$

where F_q and f_q = redimensionalized final infiltration depth and rate, respectively, as computed by the Li et al. method. For the Taylor-approximation values for total depths from (25) and (8) are

$$F_{p*}(0.122) = (0.122) + [(2)(0.122)]^{1/2} - (0.2987)(0.122)^{0.7913} = 0.559 \dots \dots \dots (35)$$

$$F_p = (0.559)(39.9) = 23.3 \text{ mm} \dots \dots \dots (36)$$

and for final rates from (22) and (11) are

$$f_{p*}(0.122) = 1 + \frac{1}{(0.559)} = 2.79 \dots \dots \dots (37)$$

$$f_p = (2.79)(5) = 13.9 \text{ mm} \dots \dots \dots (38)$$

where F_q and f_q = redimensionalized final infiltration depth and rate, respectively, as computed by the Taylor approximation.

The final rates and total depths are listed in Table 2. Note that the error produced by (24) for depth decreases as the soil texture becomes finer, while the error produced by (25) increases. In Fig. 1, the infiltration-rate and cumulative-depth curves were computed by choosing a time step of 1 min and solving the depth and rate equations of Green-Ampt, Li et al. (1976), and the approximation of the Taylor expansion of the Green-Ampt. Note that for the sandy loam, (24) underestimates the cumulative depth and slightly overestimates the infiltration rate, and that Eq. 25 follows the Green-Ampt curves closely. For the silt and silt-clay loam, both approximations give the same results.

COMMENT

If we compute the error, E_* , as in Li et al. (1976) caused by the approximation for cumulative depth as

$$E_* = 1 - \frac{F_{pred}}{F_*} \dots \dots \dots (39)$$

TABLE 2. Comparison of Final Infiltration Rates [Eqs. (1), (12), and (22)] and Total Infiltration Depth [Eqs. (3), (24), and (25)]

Soil texture (1)	Final Infiltration Rate (mm/h)			Total Infiltration Depth (mm)		
	Eq. (1) ^a (2)	Eq. (12) ^b (3)	Eq. (22) ^c (4)	Eq. (3) ^a (5)	Eq. (24) ^b (6)	Eq. (25) ^c (7)
sandy loam	31.6	32.3	31.7	42.3	39.4 (6.8) ^d	41.8 (1.1) ^d
silt	13.7	13.9	13.9	23.0	22.3 (3.0) ^d	22.3 (3.0) ^d
silty clay loam	8.3	8.4	8.5	15.1	14.8 (2.0) ^d	14.6 (3.3) ^d

^aGreen-Ampt.

^bQuadratic approximation.

^cTaylor series approximation.

^d6.8 = error computed by Eq. (26).

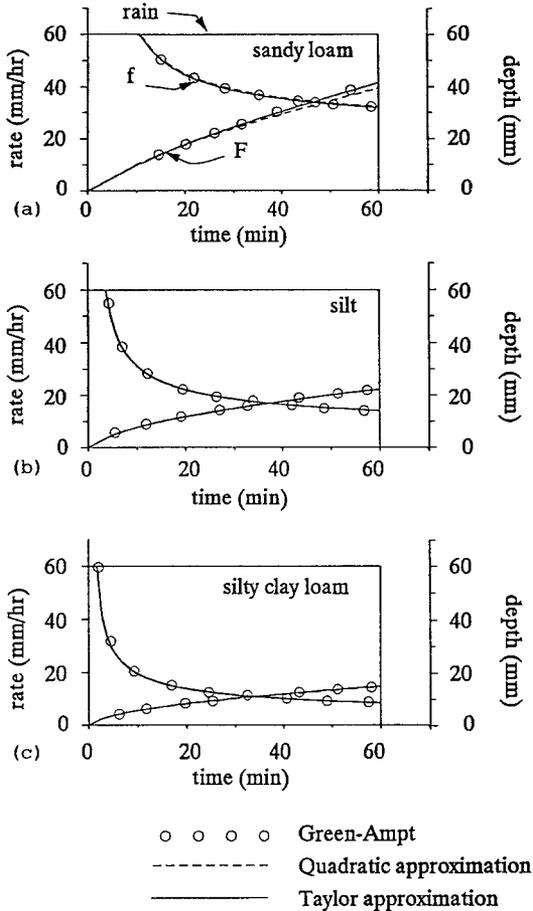


FIG. 1. Comparison of Infiltration Curves Generated by Green-Ampt, Quadratic Approximation, and Approximation of Taylor Series

where F_{pred} = cumulative depth computed from (24) or (25), we can compare the errors caused by the two approximations. In Fig. 2(a), the error is plotted as a function of F_* and, in Fig. 2(b), it is plotted as a function of t_{c*} . Eq. (25) has less error than (24) for values of F_* and t_{c*} between approximately 0.5 and 150; and the maximum error for (25) is less than that for (24). For values outside this range, (24) produces less error; however, the absolute error of both equations is small outside the range of 0.5 and 150. The reason is that small values of F_* and t_{c*} represent the cases for which (although the soil moisture is low) K_e is small and/or the duration is short, which means that the absolute amount of infiltration is small. Thus the absolute error for both equations is small. Large values of F_* and t_{c*} represent the cases for which the soil moisture is high, K_e is large, and/or the duration is long, which means that both equations tend to approach $K_e t$ asymptotically.

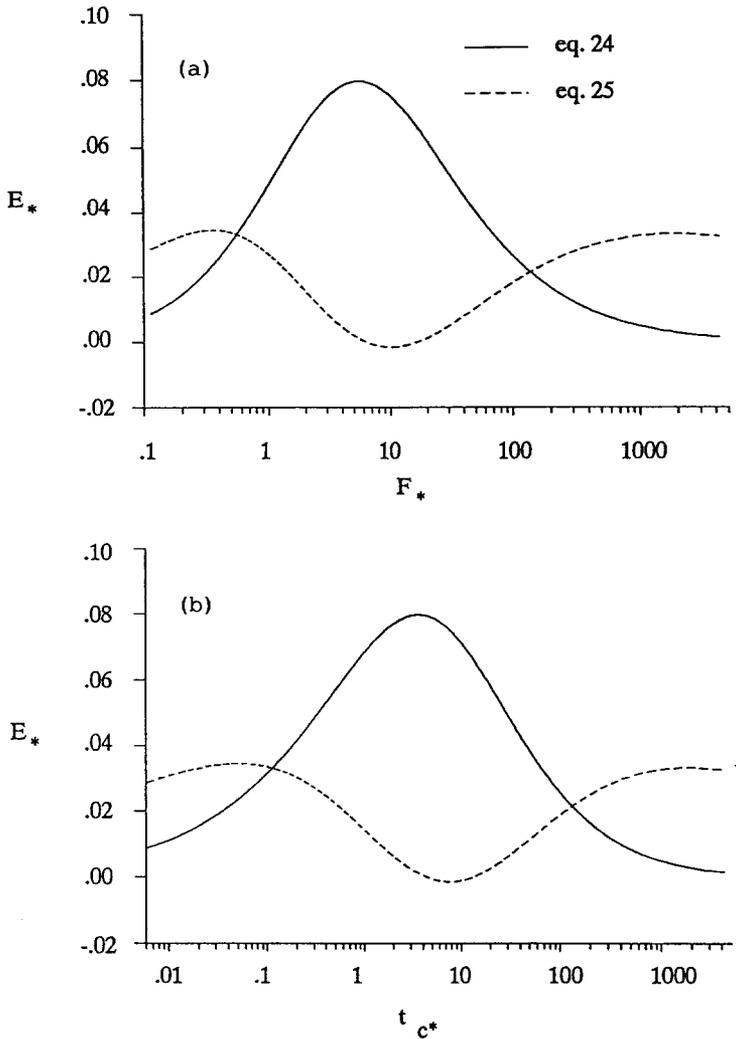


FIG. 2. Comparison of Error Caused by Eqs. (24) and (25)

SUMMARY

We developed an approximation of the Green-Ampt infiltration equation by using the first two terms in a Taylor-series expansion of the depth equation. The resulting approximation is in the form of the Philip's (1957) equation, with an added term to account for the approximation. The coefficients of the added term were fitted using nonlinear least squares regression. With the addition of the Green-Ampt Mein-Larson time correction, the approximation can be used for the case of constant rainfall with variable time to ponding. The error of the approximation was compared with the quadratic approximation of Li et al. (1976). It was shown that the new approximation has less error within the range of values of the ratio of cumulative depth to capillary potential of 0.5 to 150. This range roughly corresponds to coarser

textured soils. Outside this range both approximations result in small absolute error.

APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

- E_* = error in approximation = $1 - (F_{\text{pred}}/F_*)$;
- $F(t)$ = cumulative infiltration depth (L);
- F_p = cumulative infiltration depth computed by Li et al. (1976) (L);
- F_q = cumulative infiltration depth computed by Taylor approximation (L);
- $F_*(t_*)$ = dimensionless infiltration depth = $F(t)/N_s$;
- $F_{p*}(t_*)$ = dimensionless infiltration depth computed by (21) and (25);
- F_{pred} = dimensionless infiltration depth computed by (24) or (25);
- $F_{q*}(t_*)$ = dimensionless infiltration depth computed by (10) and (24);
- F_{tp} = cumulative infiltration depth at time to ponding (L);
- $f(t)$ = infiltration rate (L/T);
- f_p = infiltration rate computed by Li et al. (1976) (L);
- f_q = infiltration rate computed by Taylor approximation (L);
- $f_*(t_*)$ = dimensionless infiltration rate = $f(t)/K_e$;
- $f_{p*}(t_*)$ = dimensionless infiltration rate computed by (22);
- $f_{q*}(t_*)$ = dimensionless infiltration rate computed by (12);
- g = function for Taylor series expansion;
- i = rainfall rate (L/T);
- K_e = effective saturated conductivity (L/T);
- t = time (T);
- t_* = dimensionless time = $tK_e/(\psi\theta_d)$;
- t_c = time correction = $t + t_s - t_p$ (T);
- t_{c*} = dimensionless corrected time = $t_cK_e/(\psi\theta_d)$;
- t_p = time to ponding (T);
- t_s = time shift to account for infiltration at time to ponding (T);

ϵ_* = dimensionless error in Philip's equation form for Green-Ampt;
 η_e = effective porosity (L/L);
 θ_d = soil-moisture deficit (L/L);
 θ_i = initial volumetric water content (L/L); and
 ψ = average soil capillary potential (L).